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Strongly determined types

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Abstract

The notion of a *strongly determined type over A extending p* is introduced, where $p \in S(A)$. A strongly determined extension of p over A assigns, for any model $M \supseteq A$, a type $q \in S(M)$ extending p such that, if $\bar{c}$ realises q, then any elementary partial map $M \to M$ which fixes $\mathrm{acl}^{\mathrm{eq}}(A)$ pointwise is elementary over $\bar{c}$. This gives a crude notion of independence (over A) which arises very frequently. Examples are provided of many different kinds of theories with strongly determined types, and some without. We investigate a notion of multiplicity for strongly determined types with applications to 'involved' finite simple groups, and an analogue of the Finite Equivalence Relation Theorem. Lifting of strongly determined types to covers of a structure (and to symmetric extensions) is discussed, and an application to finite covers is given.

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0. Introduction

In this paper we explore a weak form of independence, suggested by forking in stability theory. We introduce the notion of a *strongly determined type* over a set A of parameters. This generalises the familiar notion of *A-definable type*. We are particularly concerned with existence questions for strongly determined types over A, where A is small (often empty).

The emphasis of the paper is on examples. In Section 1 we define strongly determined types, and discuss how it relates to familiar model-theoretic notions. Section 2 provides many examples of different kinds of theories in which strongly determined types exist. This gives evidence that there will be strongly determined types

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in most natural theories. However, in Section 3, we give some artificial constructions of ω-categorical theories with no non-algebraic strongly determined types. In Sections 4 and 5 we discuss some applications of strongly determined types. We investigate in Section 4 a version of the Finite Equivalence Relation Theorem, and discuss multiplicity, symmetry conditions on the set of strongly determined extensions of types over a set, and Hrushovski's notion of a finite simple group being *involved* in a structure. In Section 5 we give an application to covers of structures, the original motivation. We also prove in Section 5 an existence result (over a constant) for strongly determined types in smoothly approximated structures. We have tried to summarise the main thrust of the paper in Section 6. David Evans [14, Definition 2.1] has worked with a slight variation on strongly determined types, also with an application to finite covers (his notion is essentially Shelah's *non-splitting* [43]).

1. Strongly determined types

Our purpose is to generalise the following familiar condition from stability theory.

Definition. If $A \subset M$, then a type $p(\bar{x}) \in S(M)$ is *definable almost over* A if for any formula $\phi(\bar{x}, \bar{y})$ over A the set $\{\bar{b} \in M\colon \phi(\bar{x}, \bar{b}) \in p(\bar{x})\}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.

In this paper, T will denote a first-order theory over a countable language. The symbols M, N will denote models of T, which are assumed to be elementary substructures of a sufficiently saturated monster model $\mathbf{C}$. We use A, B to denote subsets of $\mathbf{C}$, assumed to be much smaller than $\mathbf{C}$. If $\bar{a}$ is a tuple, we often abuse notation by writing $\bar{a} \in M$. If no such restriction is given, then $\bar{a}$ is assumed just to live in $\mathbf{C}$. We also sometimes regard tuples as sets, and use concatenation for union (so $\bar{a}B$ might denote $\{a_1, \ldots, a_n\} \cup B$). If $r(\bar{x})$ is a type in (possibly several) variables, we denote by $r(M)$ the set of tuples from M which realise r. For any structure M and $A \subseteq M$, define $\mathrm{Aut}(M/A)$ to be the group of automorphisms of M which fix A pointwise, and $\mathrm{Aut}^o(M/A)$ to be the group of all automorphisms of M which fix setwise the classes of all A-definable finite equivalence relations on M^n for all $n \in \omega$. We call the elements of $\mathrm{Aut}^o(M/A)$ *strong automorphisms over* A (the elements of $\mathrm{Aut}^o(M) = \mathrm{Aut}^o(M/\emptyset)$ are called *strong automorphisms*). We talk similarly of *strong elementary maps over* A. It follows from the definitions that if $p(\bar{x}) \in S(M)$ is definable almost over $\emptyset$ then $\alpha(p) = p$ for every $\alpha \in \mathrm{Aut}^o(M)$. (The converse too will hold if M is saturated.)

If $\bar{c} \in \mathbf{C}$, then the *strong type* of $\bar{c}$ over A is just $\mathrm{tp}(\bar{c}/\mathrm{acl}^{\mathrm{eq}}(A))$ (this use is different from that in [14]). We write $S(\mathrm{acl}^{\mathrm{eq}}(A))$ for the set of strong types *of the sort* M over A. We shall say that M is *rich* over A if, for all $n \in \omega$, M realises all n-types of $S(\mathrm{acl}^{\mathrm{eq}}(A))$. Also, M is *very rich* over A if, for all $n \in \omega$ and all $\bar{m} \in M$, M realises all n-types from $S(\mathrm{acl}^{\mathrm{eq}}(A\bar{m}))$.

If $q(\bar{y}) \in S(\mathrm{acl}^{\mathrm{eq}}(A))$, we say that a type $p(\bar{x}, \bar{y}) \in S(A)$ is a *q-consistent $\bar{x}$-type* if for any sequence $\bar{a}_1, \ldots, \bar{a}_n$ of realisations of q the set $\bigcup\{p(\bar{x}, \bar{a}_i)\colon 1 \leqslant i \leqslant n\}$ is consistent.

A *strongly determined type* over a set A is a function ρ which assigns a q-consistent $\bar{x}$-type $\rho(q)(\bar{x},\bar{y})$ to every $q(\bar{y}) \in S(\mathrm{acl}^{\mathrm{eq}}(A))$, and which is *monotonic*: that is, if $\bar{y}'$ is a subtuple of $\bar{y}$ and $q(\bar{y}), q'(\bar{y}') \in S(\mathrm{acl}^{\mathrm{eq}}(A))$ with $q(\bar{y}) \vdash q'(\bar{y}')$, then the restriction of $\rho(q)(\bar{x},\bar{y})$ to $\bar{x}\bar{y}'$ is $\rho(q')(\bar{x},\bar{y}')$.

Suppose that ρ is a strongly determined type over A. For every B containing A define

$$\rho_B(\bar{x}) = \bigcup \{\rho(q)(\bar{x},\bar{b}):\ q \in S(\mathrm{acl}^{\mathrm{eq}}(A)), \bar{b} \models q, \bar{b} \in B\}.$$

It is easy to see that if the model M is rich over A than any strongly determined type ρ over A is determined by ρ_M. In this situation we therefore often refer to ρ_M as a strongly determined type (over A), as a *strongly determined extension* of ρ_A, or as the *M-restriction* of ρ. Observe that if M is saturated and $|A| < |M|$, then ρ_M is strongly determined if and only if any $\alpha \in \mathrm{Aut}^o(M/A)$ is elementary over any realisation of ρ_M: that is, if $\bar{c} \in M$ and $\bar{d} \models \rho_M$, then the tuples $\bar{c}$ and $\alpha(\bar{c})$ realise the same type over $A\bar{d}$. See Lemma 2.1 for another characterisation.

We begin with some easy examples. Parts (b) and (c) are subsumed by Remark 1.4 on stable theories.

Example 1.1. (a) Let $M = (\mathbf{Q}, <)$. Then the unique 1-type p over $\emptyset$ has two strongly determined extensions, namely ρ^L and ρ^R, where $\rho^L_M(x)$ holds if and only if $\forall y \in \mathbf{Q}(x < y)$, and $\rho^R_M(x)$ holds if and only if $\forall y \in \mathbf{Q}(y < x)$.

(b) Let M be a countable set with an equivalence relation with two classes, both infinite. Then the unique 1-type over $\emptyset$ has two strongly determined extensions, one corresponding to each class.

(c) Let M be a countably infinite set with an equivalence relation with infinitely many infinite classes. Then the unique 1-type over $\emptyset$ has a unique strongly determined extension ρ, where c realises ρ if and only if c is inequivalent to all members of M.

We next give an example to show that ρ_M is not necessarily definable almost over A, even when $A = \emptyset$.

Example 1.2. Take a saturated model M of $\mathrm{Th}(\Gamma, R, P_i:\ i \in \omega)$ where (Γ, R) denotes the random graph and the P_i are unary predicates picking out a family of subsets of Γ, in such a way that the P_i are random with respect to each other and R. Let ρ be a strongly determined type over $\emptyset$ extending $\bigcap\{P_i:\ i \in \omega\}$ such that $\rho_M(x)$ says that there are no R-edges between x and any element of M, and ρ' be a strongly determined type over $\emptyset$ extending $\bigcap\{P_i:\ i \in \omega\}$, such that any realisation of ρ'_M is joined by an R-edge to every element of $\bigcap\{P_i^M:\ i \in \omega\}$ and to no element of $\bigcup\{\neg P_i^M:\ i \in \omega\}$. It is clear that ρ_M is definable but ρ'_M is not.

Despite Example 1.2, there is a weak sense in which the M-restriction of a strongly determined type is definable. We say that a type $p(\bar{x}) = \mathrm{tp}(\bar{c}/M)$ is *quasidefinable almost over A* if for every $r(\bar{x},\bar{y}) \in S(\mathrm{acl}^{\mathrm{eq}}(A))$ consistent with $p(\bar{x})$ the set $r(\bar{c}, M)$ is the set of realisations in M of a complete type over $\mathrm{acl}^{\mathrm{eq}}(A)$. (Observe that since

$r(\bar{x},\bar{y})$ specifies a type in the $\bar{y}$-variables over $\mathrm{acl}^{\mathrm{eq}}(A)$, the realisations of $r(\bar{c},M)$ will realise such a type.) Superficially, this notion looks stronger than that of a strongly determined type, since if $\mathrm{tp}(\bar{c}/M)$ is quasidefinable almost over A and $\bar{b},\bar{b}'\in M$ have the same strong type over A, then $\bar{b}\bar{c}$, $\bar{b}'\bar{c}$ realise the same *strong* type over A. However, by the next lemma, under sufficient saturation the conditions are equivalent.

Lemma 1.3. *Let $p\in S(M)$. Then*
(i) *If M is rich over A and p is quasidefinable almost over A then $p=\rho_M$ for some strongly determined type ρ over A.*
(ii) *If M is very rich over A and $p=\rho_M$ for some strongly determined type ρ over A, then p is quasidefinable almost over A.*

Proof. (i) If p is quasidefinable almost over A then for any formula $\phi(\bar{x},\bar{y})$ and any $\bar{b},\bar{b}'\in M$ of the same type over $\mathrm{acl}^{\mathrm{eq}}(A)$, we have $\phi(\bar{x},\bar{b})\in p$ if and only if $\phi(\bar{x},\bar{b}')\in p$. This defines a monotonic function ρ assigning a q-consistent $\bar{x}$-type to every $q(\bar{y})\in S(\mathrm{acl}^{\mathrm{eq}}(A))$.

(ii) Suppose that $\bar{c}\models\rho_M$ and $\bar{b},\bar{b}'\in M$ realise the same type over $\mathrm{acl}^{\mathrm{eq}}(A)$. We must show that $\bar{b}\bar{c}$, $\bar{b}'\bar{c}$ realise the same types over $\mathrm{acl}^{\mathrm{eq}}(A)$. Let $E(\bar{x},\bar{y})$ (where $l(\bar{x})=l(\bar{b}\bar{c})$) be a finite equivalence relation definable over A and $\bar{b}\bar{d}\in M$ be an extension of $\bar{b}$ of the same E-class as $\bar{b}\bar{c}$. Choose $\bar{d}'\in M$ such that $\bar{b}'\bar{d}'$ has the same strong type over A as $\bar{b}\bar{d}$. Then $\mathrm{tp}(\bar{c},\bar{b}\bar{d})=\mathrm{tp}(\bar{c},\bar{b}'\bar{d}')$ (as ρ is strongly determined and $\bar{c}\models\rho_M$), so as $E(\bar{b}\bar{c},\bar{b}\bar{d})$, we have $E(\bar{b}'\bar{c},\bar{b}'\bar{d}')$. It follows by transitivity that $E(\bar{b}\bar{c},\bar{b}'\bar{c})$, as required. □

We now formulate the main notions of the paper in the most general form. We say that a theory T *has a strongly determined type over A* if there exists a non-algebraic $p\in S(A)$ which extends to a strongly determined type over A. A theory T *admits strongly determined types over A* if every type of $S(A)$ extends to a strongly determined one. Finally, T *admits strongly determined types*, if it admits strongly determined types over every set A of parameters (and for this, it suffices that T admits strongly determined types over every finite set). It is shown in Example 3.5 that a theory can admit strongly determined types over $\emptyset$ but not over some constant expansion.

Remark 1.4. Strongly determined types offer a weak notion of independence: if $A\subset B$, then $\bar{c}$ is independent from B over A if $\mathrm{tp}(\bar{c}/A)$ has a strongly determined extension ρ such that $\rho_B=\mathrm{tp}(\bar{c}/B)$. This notion generalises independence in stable theories. For let T be a stable complete theory over a countable language, and $M\models T$. Then by the finite equivalence relation theorem and forking symmetry, if $A\subset M$ and $p\in S(M)$ does not fork over A, then p realises a strongly determined type over A. Hence, in the stable case,
(a) every type over M realises a strongly determined type over some countable subset of M,

(b) if $p(\bar{x})$ is a complete type over A, then p extends to a strongly determined type over A.

In fact, by Lemma 1.5(iv), in stable theories strongly determined extensions correspond exactly to global non-forking extensions.

We record without proof four trivial facts.

Lemma 1.5. *Let $A \subset M$.*

(i) *If r is a n-type over A with strongly determined extensions, and p is a restriction of r to a subset of the arguments, then p has strongly determined extensions.*

(ii) *Any algebraic type over A has a strongly determined extension.*

(iii) *If $A \subset B \subset M$ where M is rich over B, and ρ is strongly determined over A, then ρ_M is a strongly determined extension of ρ_B.*

(iv) *If M is very rich over A and ρ is strongly determined over A, then ρ_M does not divide over A (in the sense of* [43, p. 85]).

A simple counting argument, which we omit, gives the following bound on the number of strongly determined extensions of a type over A.

Lemma 1.6. *Suppose that the underlying language L has size λ, and that the parameter set A has size κ. Then if p is a type (of our theory T) over A, there are at most $2^{2^{\max\{\lambda,\kappa\}}}$ strongly determined extensions of p.*

We make some remarks on *simple* theories, introduced in [42] and developed in [30, 29]. First, Example 3.2 below shows that simple theories (even when ω-categorical) may not have strongly determined types over $\emptyset$. On the other hand, if a simple theory admits strongly determined types then by Lemma 1.5(iv), the independence provided by strongly determined types is stronger than that provided by forking: that is, if p divides over A then p does not extend to a strongly determined type over A.

Proposition 1.7. *Let T be a countable complete theory.*

(i) *Suppose that for any set B, any $p \in S_1(B)$ is quasidefinable almost over a countable subset of B. Then T is stable.*

(ii) *Suppose that T is simple, and for any $A \subseteq B$ and $p \in S_1(B)$, p does not fork over A if and only if p is strongly determined over A. Then T is stable.*

Proof. (i) Let M be an $\aleph_1$-saturated model of T of cardinality at least $2^{2^{\aleph_0}}$ with $|M| = |M|^{\aleph_0}$. By Lemma 1.6, for any countable $A \subset M$ the number of types quasidefinable almost over A is not greater than $2^{2^{\aleph_0}}$. This yields that the number of types over M equals the cardinality of M.

(ii) Since T is simple, for every $p \in S(M)$ there is a countable $A \subset M$ such that p does not fork over A (see [30, Fact 1.6(2)]). This means that any $p \in S(M)$ is quasidefinable almost over a countable subset of M, so (i) is applicable. $\square$

We emphasise that unlike forking in simple theories, it is not true in general that every type over a model is strongly determined over a small subset. Also, we do not in general have anything resembling forking symmetry (but see Section 4.2 for more on this). For example, let $M=(\mathbf{Q},<)$. Let ρ be the strongly determined type over $\emptyset$ such that if $c \models \rho_M$ then $c<x$ for all $x \in M$. Then if $c \models \rho_M$ and $d \models \rho_{Mc}$, then $c \not\models \rho_{Md}$.

In the case when T is ω-categorical and A is finite, strongly determined types over A are familiar. For example, in this case a type quasidefinable almost over A is definable (for on any A-definable set there is a finest A-definable finite equivalence relation). Moreover, for every strongly determined type ρ and any finite $B \supseteq A$, the type ρ_B is defined by a single formula.

The notion of strongly determined type over A is analogous to that of *special extension* [33, Definition 2.19]. The difference is that in Lascar's setting, the set A is always a model. He shows, for example (Proposition 2.23) that if $M \subset B$ and ρ is a strongly determined type over M, then ρ_B is a coheir of ρ_M.

In Definition I.2.6 of [43], Shelah defines non-splitting extensions as follows: for $A \subset B$, a type $p(\bar{x})=\mathrm{tp}(\bar{d}/B)$ *does not split over* A if any $\bar{b}$ and $\bar{b}'$ of the same type over A have the same type over $A\bar{d}$. It is clear that for $A \subset M$ with $\mathrm{acl}^{\mathrm{eq}}(A)=\mathrm{dcl}^{\mathrm{eq}}(A)$, if ρ is a strongly determined type over A then ρ_M does not split over A. On the other hand, if M is rich over A and $p \in S(M)$ does not split over A, then there is a strongly determined type over A which extends p (this is Exercise I.2.3 of [43]). Shelah shows that forking can be replaced by splitting in several places in stability theory (for example, see Lemmas I.2.5 and I.2.6 of [43]). However, non-splitting does not allow the standard development of multiplicity: even in the stable case a non-forking extension can split over the base set. Shelah's notion of *non strongly splitting extension* [43, Definition III.1.2] is essentially the same as that of a strongly determined extension. As we have already noted, for stable theories strongly determined extensions over A correspond exactly to global non-forking extensions of types over A. This is very close to Shelah's Theorem III.1.6.

Let ρ be a strongly determined type over A, and suppose that $A \subset M \subset B$. It is natural to think of ρ_B as an heir of ρ_M (see [35]). However, Example 1.2 shows that, unlike heirs in stable theories, ρ_B is not necessarily determined by ρ_M. Indeed, if in Example 1.2, N is an elementary substructure of M with $\bigcap\{P_i^N\colon i \in \omega\}=\emptyset$ then ρ_N and ρ'_N are the same (here, $A=\emptyset$). Also, ρ'_M is not an heir of ρ'_N in the usual sense. Nevertheless, we can adapt the notion of *heir*.

Let $p(\bar{x}) \in S(M)$ and $A \subset M \subset B$, and suppose that M is very rich over A. If $p'(\bar{x}) \in S(B)$ extends p, we say that $p'(\bar{x})$ is an *s-heir of* p *over* A if for every $\bar{m} \in M$, every formula $\phi(\bar{m},\bar{x},\bar{y})$ and every $q(\bar{y}) \in S(\mathrm{acl}^{\mathrm{eq}}(A\bar{m}))$, we have: if there is $\bar{b}' \in q(B)$ satisfying $\phi(\bar{m},\bar{x},\bar{b}') \in p'$, then there is $\bar{b} \in q(M)$ satisfying $\phi(\bar{m},\bar{x},\bar{b}) \in p$. The usual compactness argument [41, Lemma 1.15] shows that each $p(\bar{x}) \in S(M)$ extends to an s-heir from $S(B)$ over A. Furthermore, any s-heir of $p \in S(M)$ over A is an heir of p (for by the richness assumption, each set $q(M)$ is non-empty).

With A,M,B as above, if ρ is a strongly determined type over A then ρ_B is an s-heir of ρ_M over A. The following lemma shows that this s-heir is unique. The proof is easy.

Lemma 1.8. *Let $A \subset M \subset B$ and suppose that M is very rich over A. Let $p \in S(M)$ be a type quasidefinable almost over A. Then p has a unique s-heir over A in $S(B)$.*

The converse of the lemma is not true. For example, take the random graph Γ and an element $b \in \Gamma$. Let $c \notin \Gamma$ be adjacent to b and not adjacent to the elements of $\Gamma \setminus \{b\}$. The type $\mathrm{tp}(c/\Gamma)$ is not quasidefinable almost over $\emptyset$ but for any $B \supseteq \Gamma$, it has a unique s-heir over $\emptyset$ in $S(B)$.

2. Examples with strongly determined types

We collect in this section some different kinds of theories with a rich supply of strongly determined types. We begin with two existence criteria for strongly determined types.

Lemma 2.1. *Let $A \subset M$, and suppose that M is very rich over A. Let $p(\bar{x}) \in S(A)$. Then the following are equivalent:*
(i) *p extends to a strongly determined type over A;*
(ii) *for every finite set Γ of finite partial strong elementary maps $M \to M$ over A, there exists $\bar{c} \in p(M)$ such that all the maps in Γ are elementary over $\bar{c}$.*

Proof. The direction (i) $\Rightarrow$ (ii) is clear. For (ii) $\Rightarrow$ (i), extend the elementary diagram of M by $p(\bar{c})$ and the sentences of the following form: $\phi(\bar{c}, \bar{m}) \leftrightarrow \phi(\bar{c}, \bar{m}')$ where $\bar{c}$ is a tuple of new constants and $\bar{m}, \bar{m}' \in M$ have the same strong type over A. By compactness p extends to a strongly determined type over A. □

Lemma 2.2. *Let T be a complete theory, and suppose that for all $M \models T$ and all $r > 0$, and all $\bar{a} \in M^r$, every 1-type of $\mathrm{Th}(M)$ over $\bar{a}$ extends to a strongly determined type over $\bar{a}$. Then for all $\bar{a} \in M^r$ and all $n > 0$, every n-type of $\mathrm{Th}(M)$ over $\bar{a}$ has a strongly determined extension over $\bar{a}$.*

Proof. Pick M and $\bar{a} \in M$, and let p be an n-type of $\mathrm{Th}(M)$ over $\bar{a}$. Form an increasing chain of ω-saturated structures, $M := M_0 \prec M_1 \prec \cdots \prec M_n$, together with $c_1, \ldots, c_n$ such that for each $i < n$, if $p \nvdash x_i \in \mathrm{acl}(\bar{a}\{x_j : j < i\})$ then $c_i \in M_i \setminus M_{i-1}$ and c_i realises a 1-type over M_{i-1} quasidefinable almost over $(c_1 \ldots c_{i-1}, \bar{a})$, and such that $(c_1, \ldots, c_n)$ realises p. Using Lemma 1.3, it follows by induction that for each i, $(c_1, \ldots, c_i)$ realises a strongly determined type over $\bar{a}$. □

2.1. *Strongly determined types obtained from algebraic closure*

The following result provides some algebraic examples which have strongly determined types. They will not be ω-categorical.

Proposition 2.3. *Suppose that $A \subset M$. Let $p(\bar{x}) \in S(A)$, and suppose that every formula of $p(\bar{x})$ is realised in* $\mathrm{acl}(A)$. *Then $p(\bar{x})$ extends to a strongly determined type over A.*

Proof. We may suppose that M is rich over A. Extend the elementary diagram of M by adjoining $p(\bar{c})$ and all sentences of the form $\phi(\bar{c}, \bar{m}) \leftrightarrow \phi(\bar{c}, \bar{m}')$, where $\bar{m}, \bar{m}' \in M$ have the same strong types over A. This set is consistent (interpret the tuple $\bar{c}$ by elements of $\mathrm{acl}(A)$). It follows by compactness that p extends to a strongly determined type over A. □

Corollary 2.4. *If* $\mathrm{acl}(A)$ *is a model of* $\mathrm{Th}(M, a)_{a \in A}$, *then* $\mathrm{Th}(M)$ *admits strongly determined types over A. In particular, if T has an infinite model, has a constant symbol, and has definable Skolem functions, then it admits strongly determined types.*

Proof. Replace M by a rich elementary extension of $\mathrm{acl}(A)$, and apply Proposition 2.3. □

Example 2.5. By Corollary 2.4, models of PA admit strongly determined types. The same also holds for $\mathbf{Q}_p$ [11, Theorem 3.2]. Also, $\mathbf{Q}_p$ (or $\mathbf{Z}_p$) endowed with subanalytic structure as in [10] admit strongly determined types, since both structures are algebraic over $\emptyset$.

2.2. Notions of minimality and strongly determined types

A totally ordered structure $(M, <, \ldots)$ is said to have *weakly o-minimal* theory if, for every $(N, <, \ldots)$ elementarily equivalent to M, every definable subset of N is a finite union of convex sets. See [39] for examples and some structure theory.

Theorem 2.6. *Let T be a weakly o-minimal theory. Then T admits strongly determined types.*

Proof. For convenience, we shall suppose that the underlying order is dense, although this is not necessary. By incorporating any set A of parameters into the theory, and applying Lemma 2.2, it suffices to show that any non-algebraic 1-type p over $\emptyset$ has a strongly determined extension. Let $M \models T$ be ω-saturated. The set P of realisations of p in M is convex, and without greatest or least elements. We have $M = L \cup P \cup R$, (a disjoint union), where $L := \{x \in M\colon \forall y \in P(x < y)\}$, and $R := \{x \in M\colon \forall y \in P(y < x)\}$. For convenience, we suppose that both L and R are non-empty. Let C_1 be the cut $(L, P \cup R)$ and C_2 be the cut $(L \cup P, R)$. We regard C_1 and C_2 as non-algebraic partial types over M. Clearly, if there is a strongly determined (over $\emptyset$) extension ρ of p, then ρ_M has realisations in just one of C_1, C_2. We will only consider realisations in C_1.

We use the analysis of types over M given in [31], though our argument is self-contained. First, if L has no greatest element and is not definable in M, then C_1

extends to a unique type over M, containing all formulas over M which are satisfied in an interval (c,d) of M where $c \in L$ and $d \in P$. If L has no greatest element and is definable, by a formula $\chi(x,\bar{m})$, say, then C_1 has two extensions over M, one consisting of all formulas satisfied by a final segment of L, the other consisting of all formulas satisfied by an initial segment of P. Finally, if L has a greatest element, b say, then C_1 extends to a unique type over M, namely the type consisting of all formulas $\phi(x,\bar{m})$ which contain a non-empty interval (b,c) of M.

We shall show that all of these extensions of C_1 have strongly determined extensions over $\emptyset$. So fix $\bar{d},\bar{d}' \in M$ with the same type over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, and let $\psi(x,\bar{y})$ be a formula with $l(\bar{y}) = l(\bar{d})$. We may suppose that $\psi(x,\bar{y})$ always defines a convex set. By compactness and ω-saturation, $\psi(x,\bar{d})$ contains an initial segment of P if and only if $\psi(x,\bar{d}')$ does: for if there are $b_1 < b_2$ in P such that $\neg\psi(b_1,\bar{d}) \wedge \psi(b_2,\bar{d})$, then, as $\bar{d},\bar{d}'$ have the same type over $\emptyset$, it is consistent that there are $b_1' < b_2'$ in P such that $\neg\psi(b_1',\bar{d}') \wedge \psi(b_2',\bar{d}')$, so by ω-saturation, such b_1', b_2' exist. Likewise, if L has no greatest element, $\psi(x,\bar{d})$ contains a final segment of L is and only if $\psi(x,\bar{d}')$ does. It follows from this and the above analysis of types over M that any of the (at most two) extensions of C_1 over M has a strongly determined extension over $\emptyset$. $\square$

We turn next to the notions of minimality (such as *C-minimality*), introduced in [40] and developed in [19, 20].

Definition 2.7. Suppose that $L \subset L^+$ are languages, and M is an L^+-structure. We say that M is *L-minimal* if for every parameter-free L^--formula $\phi(x,\bar{y})$ there is an L-formula $\psi(x,\bar{z})$ without quantifiers or parameters such that for all $\bar{a} \in M$ there is $\bar{b} \in M$ such that

$$M \models \forall x(\phi(x,\bar{a}) \leftrightarrow \psi(x,\bar{b})).$$

Note that $\bar{y}$ and $\bar{z}$ above may have different lengths. This notion yields strong minimality if L is empty (or, strictly speaking, just has the equality symbol), o-minimality if L just has a binary relation interpreted as a total order on M, C-minimality if L just has a ternary relation satisfying the C-relation axioms described in [40, 19], and P-minimality if L is as in [20]. Observe that if (in the above notation) M is L-minimal and $A \subset M$ then M as an $L^+(A)$-structure is L-minimal (so also $L(A)$-minimal).

Lemma 2.8. *Suppose that $L \subset L^+$ and that the L^+-structure M is L-minimal. Assume that for every L^+-structure N which is elementarily equivalent to M, there is $c \notin \mathrm{acl}(\emptyset)$ such that whenever $\bar{a},\bar{a}' \in N$ have the same strong L^+-type over $\emptyset$, the tuples $\bar{a}c$ and $\bar{a}'c$ have the same quantifier-free L-type over $\emptyset$. Then* $\mathrm{Th}(M)$ *has a strongly determined 1-type over $\emptyset$.*

Proof. We may suppose that M is very rich over $\emptyset$. Choose c as in the condition of the lemma (with $N = M$). Let $\bar{b},\bar{b}'$ have the same strong L^+-type over $\emptyset$. Let $\phi(x,\bar{y})$ be an L^--formula (with $l(\bar{y}) = l(\bar{b})$), and suppose that $\phi(c,\bar{b})$ holds. There is a quantifier-free

L-formula $\psi(x,\bar{z})$ such that

$$M \models \forall \bar{y} \exists \bar{z}\, \forall x\, (\phi(x,\bar{y}) \leftrightarrow \psi(x,\bar{z})).$$

Choose $\bar{a} \in M$ such that $M \models \forall x(\phi(x,\bar{b}) \leftrightarrow \psi(x,\bar{a}))$. Then $M \models \psi(c,\bar{a})$. As M is very rich over $\emptyset$, there is $\bar{a}'$ such that $\bar{b}\bar{a}$ and $\bar{b}'\bar{a}'$ have the same strong L^+-types over $\emptyset$. Then by the assumption on c, we have $\psi(c,\bar{a}')$, so $\phi(c,\bar{b}')$, as required. □

The following variant gives a criterion for admitting strongly determined types. It is harder to apply, since one first needs a good understanding of 1-types.

Lemma 2.9. *Suppose that $L \subset L^+$ and that the L^+-structure M is L-minimal. Also suppose that for every N elementarily equivalent to M (as L^+-structures) and every finite $A \subset N$ and* 1*-type p in $L^+(A)$, there is c realising p such that for every $\bar{b}, \bar{b}' \in N$ with the same strong L^+-type over A, $\bar{b}c$ and $\bar{b}'c$ satisfy the same quantifier-free L-formulas over A. Then* Th(M) *admits strongly determined types.*

Proof. This is similar to the proof of Lemma 2.8. By Lemma 2.2, it suffices to show that 1-types (in L^+) over $\emptyset$ have strongly determined extensions. □

Theorem 2.10. *Any P-minimal or C-minimal theory has a strongly determined* 1*-type.*

Proof. These follow from Lemma 2.8. In the P-minimal case, we just use Example 2.5.

Suppose that M is C-minimal, and rich. We adopt the terminology of [40, 19], and just sketch the proof. The easiest case is when $M \models \forall y \forall z \exists x C(x; y, z)$. In this case, simply choose c so that for any $x, y \in M$ we have $C(c; x, y)$. The next easiest case is when M is the union of infinitely many disjoint cones all at the same 'node'. In this case, choose c in any new cone not meeting M. Finally, suppose that M is the union of finitely many disjoint cones. Observe that there is an L^+-definable finite equivalence relation over $\emptyset$ whose classes are these cones. Choose one of these cones and iterate the above argument (ω many times if necessary) in the cone until we find c. □

Remark 2.11. (1) In fact, the last argument can be extended, using Lemma 2.9, to show that C-minimal structures admit strongly determined types. This requires analysing 1-types in C-minimal structures, and we omit the details.

(2) The known examples of P-minimal structures are p-adically closed fields, models of the theory of $\mathbf{Q}_p$ with subanalytic structure (by [12]), and reducts of the latter which expand the former. Some examples of C-minimal structures are given in [40]. In particular, algebraically closed valued fields (with a C-relation naturally defined) are C-minimal, as are their expansions by a certain ring of subanalytic functions (by [36]).

(3) Lemma 2.8 can be used also to prove that o-minimal structures have strongly determined types. This of course follows from Theorem 2.6.

2.3. Binary homogeneous structures

If L is a relational language, then an L-structure is said to be *homogeneous* if its domain is countably infinite and every isomorphism between finite substructures extends to an automorphism. We use *homogeneous* in this sense throughout the paper – it is not quite the standard model-theoretic meaning, which says just that finite elementary maps (rather than isomorphisms) extend. Much of our original motivation came from questions about homogeneous and ω-categorical structures (see also Section 4).

Proposition 2.12. *Let M be homogeneous over a finite binary relational language, suppose that $\mathrm{acl}(B) = B$ for all $B \subset M$, and let $a \in M$. Then $\mathrm{tp}(a/\emptyset)$ extends to a strongly determined type over a.*

Proof. Let $\Sigma(x)$ be the set

$$\{x \neq m \colon m \in M\} \cup \{\psi(x, \bar{m}) \leftrightarrow \psi(a, \bar{m}) \colon \bar{m} \in M, a \notin \bar{m}\}.$$

This extends to the required type. □

We remark that there is no known binary homogeneous structure M with *primitive* automorphism group such that $\mathrm{acl}(A) \neq A$ for some $A \subset M$. Indeed, Cherlin [7] has conjectured that there are none. Also, the last result suggests the following question. By Theorem 3.3, it has a negative answer if the assumption 'binary' is deleted.

Question: If M is homogeneous over a finite binary relational language is there always a finite set $A \subset M$ such that M has a non-algebraic strongly determined type over A (or even admits strongly determined types over A)?

2.4. The joint embedding property

Let T be a complete first-order theory and $M \models T$. We say that M satisfies the *joint embedding property with respect to strong maps* if for any tuples $(\delta_1, \ldots, \delta_n)$ and $(\gamma_1, \ldots, \gamma_n)$ of finite strong elementary partial maps $M \to M$ there exists an elementary partial map α such that $\mathrm{dom}(\alpha) \supset \mathrm{dom}(\gamma_i) \cup \mathrm{ran}(\gamma_i)$ (for $i = 1, \ldots, n$) and each $\delta_i \cup \gamma_i^{\alpha}$ is elementary. In practice, we only use (and need) this condition when each γ_i is the identity on its domain. By the following result, this condition provides strongly determined types, even under an assumption weaker than richness.

Proposition 2.13. *Suppose that M has the joint embedding property with respect to strong maps. Assume that for any type $q(\bar{y}) \in S(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$, each formula $\psi(\bar{y}) \in q$ has a realisation $\bar{a} \in M$ with the following property: if $\bar{a}', \bar{a}''$ are subtuples of $\bar{a}$, and $q(\bar{y})$ implies that the corresponding $\bar{y}'$ and $\bar{y}''$ have the same strong types, then $\bar{a}', \bar{a}''$ have the same strong types. Then $\mathrm{Th}(M)$ admits strongly determined types over $\emptyset$. In particular, if M is rich over $\emptyset$ and satisfies the joint embedding property with respect to strong maps then $\mathrm{Th}(M)$ admits strongly determined types over $\emptyset$.*

Proof. We use Lemma 2.1. Let $p(\bar{x}) \in S(\emptyset)$ and B be a finite subset of an ω_1-saturated model M' with $M \preccurlyeq M'$. We want to extend p to some p' over B such that the tuples from B of the same strong type have the same type over any realisation of p'. Let $q(\bar{y})$ be the strong type of some enumeration of B. Let Σ be the set consisting of $p(\bar{x}) \cup q(\bar{y})$ together with all formulas of the following form, where $\bar{y}'$ and $\bar{y}''$ correspond to tuples from B of the same strong type:

$$\phi(\bar{x}, \bar{y}') \leftrightarrow \phi(\bar{x}, \bar{y}'').$$

It suffices to show that Σ is consistent. So let Σ' be a finite subset of Σ. Choose $\bar{a}$ realising $\Sigma' \cap q(\bar{y})$ in M such that for any two tuples of elements from $\bar{a}$, they have the same strong type if $q(\bar{y})$ implies that the corresponding subtuples of $\bar{y}$ have the same strong type. It follows from the joint embedding property that there is $\bar{b}$ realising $\Sigma' \cap p(\bar{x})$ in M such that the tuple $\bar{a}\bar{b}$ realises Σ' in M. □

This provides a rich class of structures admitting strongly determined types. We mention several settings where it is applicable.

Let M be countable. Consider $G = \mathrm{Aut}(M)$ as a complete metric space by defining $d(g,h) = 2^{-n}$, where n is least such that $g(x_n) \neq h(x_n)$ or $g^{-1}(x_n) \neq h^{-1}(x_n)$ (here $\{x_n\colon n \in \omega\}$ is a fixed enumeration of M). A tuple $(g_1, \ldots, g_m) \in G^m$ is *generic* if its conjugacy class is comeagre in G^m in the product topology. It is shown in [46] (see the proof of Theorem 2.2) that if M is homogeneous then the existence of a generic tuple in G^m implies the joint embedding property with respect to m-tuples of partial maps. In particular, we have:

Corollary 2.14. *If M is countable and saturated, $G := \mathrm{Aut}(M)$, and all G^m $(m \in \omega)$ have generic tuples then* $\mathrm{Th}(M)$ *admits strongly determined types over $\emptyset$.*

Examples of such structures can be found in [23]. It is worth noting that the existence of generic tuples is much stronger than just the joint embedding property with respect to strong maps (see [27] for a complete characterisation of the existence of generics).

The joint embedding property with respect to strong maps also holds in many finitely homogeneous structures. For example, it holds for any M with the age having the *nice amalgamation property*, defined next.

Let L be a countable relational language, and K a class of finite L-structures. We say that K has the *nice amalgamation property* (NAP) if, given $A, B_1, B_2 \in K$ and embeddings $f_i : A \to B_i$, there is $C \in K$ containing B_1, and an embedding $h : B_2 \to C$, such that $h(f_2(x)) = f_1(x)$ for all $x \in A$, $h(B_2) \cap B_1 = f_1(A)$, and no tuple of $B_1 \cup h(B_2)$ which satisfies a relation of L meets both $h(B_2) \backslash B_1$ and $B_1 \backslash h(B_2)$.

Proposition 2.15. *Let M be a homogeneous structure over a countable relational language L, and suppose that the class of finite structures which embed in M has NAP. Then the joint embedding property with respect to strong maps holds for M, over any set of parameters. In particular, M admits strongly determined types.*

Proof. Easy. □

As an example of the above proposition, the theory of the random graph has exactly two strongly determined 1-types over $\emptyset$, corresponding to adjacency and non-adjacency (strictly, the former follows by applying the lemma to the complement of the graph). Likewise, for any of the $2^{\aleph_0}$ non-isomorphic countable homogeneous digraphs constructed by Henson in [21], the unique 1-type over $\emptyset$ has a strongly determined extension (again given by non-adjacency).

Many other ω-categorical structures have the joint embedding property with respect to strong maps, even though this does not follow from Proposition 2.15. This, for example, yields that the countable universal homogeneous partial order, the countable universal homogeneous distributive lattice, and the countable atomless boolean algebra, each have exactly three strongly determined types over $\emptyset$ extending the unique non-algebraic 1-type. If $(P, \leqslant)$ is one of the countable doubly homogeneous semilinear orders classified by Droste in [13], then the unique 1-type over $\emptyset$ has precisely two strongly determined extensions (corresponding to being less than everything in P or to being incomparable to everything in P). For the (ternary) C-relations derived from these semilinear orders, the unique 1-type has precisely one strongly determined extension. These all follow easily from Lemma 2.1.

The case of atomless boolean algebras can also be handled by the following proposition where the idea of the joint embedding property for strong maps still works. See [22] for background.

Proposition 2.16. *Let T^* be a model completion of a universal Horn theory T. Then T^* admits strongly determined types over $\emptyset$.*

Proof. Let $M \models T^*$ be ω_1-saturated. Let $A, B \subset M$ be finite and $\delta_1, \ldots, \delta_n$ and $\gamma_1, \ldots, \gamma_n$ be partial strong maps $A \to A$ and $B \to B$ respectively. Since M is ω_1-saturated, there are countable substructures $\hat{A}, \hat{B} \subset M$ and $\hat{\delta}_i \in \mathrm{Aut}(\hat{A})$ and $\hat{\gamma}_i \in \mathrm{Aut}(\hat{B})$ (for $1 \leqslant i \leqslant n$) such that $A \subset \hat{A}, B \subset \hat{B}$ and every $\hat{\delta}_i$ (respectively, $\hat{\gamma}_i$) extends δ_i (respectively, γ_i). Let C be a direct product of $\hat{A}$ and $\hat{B}$. Then any $\hat{\delta}_i \cup \hat{\gamma}_i$ induces an automorphism of C. By ω_1-saturation there is an embedding α of C into M over $\hat{A}$. Since T^* has elimination of quantifiers, any $\delta_i \cup \gamma_i^{\alpha}$ is elementary in M. Now apply Lemma 2.1. □

This proposition is applicable in the case of the variety of rings of characteristic p satisfying $x^{p^n} = x$. By remarks in [2, p. 27], this theory has a model completion, whose countable model is the boolean power (by the countable atomless boolean algebra) of the field with p^n elements.

The authors do not know if the theories of the following structures have strongly determined types over $\emptyset$:

(1) the countable universal locally finite group G_{PH}, in which any isomorphism between finite subgroups is induced by conjugation in G_{PH} [18];
(2) the group $FS(\omega)$ of all finitary permutations of ω;
(3) the free group $\mathbf{F}_{\omega}$ of rank ω.

All these structures have the joint embedding property with respect to strong maps (in fact, it follows from the methods used in [3] that $\mathrm{Aut}(\mathbf{F}_\omega)$ has generic tuples of arbitrary length). It is quite easy to see that if G is one of the first two groups then any type over $\emptyset$ has an extension over G which is quasidefinable almost over $\emptyset$. The problem is that none of these groups is rich over $\emptyset$, and there is no satisfactory description of their elementary extensions which are rich over $\emptyset$.

It seems possible that some variant of Proposition 2.13 will be useful in the case of the free group. Indeed, since any pair of automorphisms of $\mathbf{F}_\omega$ can be amalgamated to an automorphism of the elementary supermodel $\mathbf{F}_\omega * \mathbf{F}_\omega$, it follows that any automorphism of $\mathbf{F}_\omega$ is strong. Now applying the proof of Proposition 2.13 one can easily show that if $\mathrm{Th}(\mathbf{F}_\omega)$ does not admit strongly determined types over $\emptyset$ then there are a type $q(\bar{y}) \in S(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ and a formula $\psi(\bar{y}) \in q$ such that any $\bar{a} \in \psi(\mathbf{F}_\omega)$ has subtuples $\bar{a}'$ and $\bar{a}''$ not in the same orbit of $\mathrm{Aut}(\mathbf{F}_\omega)$, where the corresponding $\bar{y}'$ and $\bar{y}''$ have the same strong type in $q(\bar{y})$.

On the other hand, Proposition 2.13 is useless in the case of Hall's group G_{PH}. Indeed, by a compactness argument there is an elementary extension G^* and $a, b \in G^*$ of the same strong type over $\emptyset$ which are not conjugate in G^*. Then $\neg\exists z(y_1 = y_2^z)$ works as a counterexample for ψ in the proposition.

3. Examples without strongly determined types

The results of the last section suggest that all ω-categorical structures admit strongly determined types, at least after a finite number of elements have been named. In this section we show that this is not true.

First, we describe here some (hopefully representative) examples of ω-categorical structures without strongly determined types over $\emptyset$.

Example 3.1. Let (M, C) be the unique countable dense circular order (so C is a ternary relation). Then there is no strongly determined 1-type over $\emptyset$. For if $a \neq b$ then (a,b) and (b,a) have the same strong type over $\emptyset$, but for any $c \in M$, $\mathrm{tp}(a,b,c) \neq \mathrm{tp}(a,c,b)$.

Example 3.2. Let (M, R) be the universal homogeneous two-graph [4, p. 47; 44], that is, the ternary relational structure whose domain is the random graph, where a triple of vertices satisfies R if and only if it contains 1 or 3 edges. Then M does not have strongly determined 1-types over $\emptyset$. For let a realise such a type over M. Then either
(a) for all distinct $x, y \in M$, $Rxya$ holds, or
(b) for all distinct $x, y \in M$, $\neg Rxya$ holds.
In the first case, pick $x, y, z \in M$ such that $\neg Rxyz$. Then an odd number of triples from x, y, z, a satisfy R, a contradiction. In case (b), pick $x, y, z \in M$ with $Rxyz$, and argue similarly. Note that since this structure is a reduct of the random graph, its theory is supersimple of SU-rank one (in the sense of [29]).

These examples show that the property of having strongly determined 1-types over $\emptyset$ is not preserved under reducts. Another example is provided by the D-relation derived from any countable 2-homogeneous semilinear order (see [13] for the classification of countable 2-homogeneous semilinear orders, and [1] for more on D-relations).

From the above examples, more algebraic examples can be manufactured. For example, let F be an ordered field, and endow the projective line $\mathrm{PG}(1,F)$ with a circular order K by stereographic projection. We could add structure given either by cross-ratio (a relation of arity 4 for each value of the cross ratio) or by adjoining a relation symbol for each orbit of the action of $\mathrm{PSL}(2,F)$, which preserves the circular order and is 2-transitive. As in Example 3.1, the 1-type over $\emptyset$ has no strongly determined extension. Likewise, if F is a valued field, one can define a D-relation on $\mathrm{PG}(1,F)$ as in [38, p. 105], and, arguing as in the last paragraph, obtain a natural structure on $\mathrm{PG}(1,F)$ where the 1-type over $\emptyset$ has no strongly determined extension.

In the above examples of homogeneous circular orders, two-graphs, and D-relations, after finitely many (in fact one) constants are added to the language, a structure is obtained which admits strongly determined types over $\emptyset$. We now give an example of a finitely homogeneous structure (not binary) such that for any finite set of parameters A, no non-algebraic type over A extends to a strongly determined type.

Theorem 3.3. *There is a structure M, homogeneous in a finite relational language, such that for any finite $C \subset M$, the structure M does not have any strongly determined types over C.*

Proof. Let L be a language with two ternary relations S and T, and a quaternary relation Q. A sequence $(a_1,\dots,a_n,b)$ is a *good cycle* if $n \geqslant 13$

(a) Sba_ia_j holds if and only if $j \in \{i+1, i-1\}$ (considered mod n),

(b) for any distinct $x,y \in \{b,a_1,\dots,a_n\}$ there is $z \in \{b,a_1,\dots,a_n\}\setminus\{x,y\}$ such that $\neg Sxyz$ holds.

The *start* of the above good cycle is the element b. In a good cycle $(a_1,\dots,a_n,b)$, a *successive pair* is a pair (a_i,a_{i+1}) or (b,a_1) or (a_n,b). A *successive triple* is an ordered 3-set (x,y,z) where (x,y), (y,z) are successive pairs and $x \neq z$.

We consider the following collection Σ of axioms.

1. $Sxyz \vee Txyz \rightarrow (x \neq y \wedge x \neq z \wedge y \neq z)$.
2. $Qxyzw \rightarrow x,y,z,w$ are distinct.
3. $Sxyz \rightarrow (Sxzy \wedge \neg Syzx \wedge \neg Szyx)$.
4. $Tx_1x_2x_3 \rightarrow \forall \pi \in S_3(Tx_{1\pi}x_{2\pi}x_{3\pi} \leftrightarrow \pi \in A_3)$.
5. $Qx_1x_2x_3x_4 \rightarrow \forall \pi \in S_4(Qx_{1\pi}x_{2\pi}x_{3\pi}x_{4\pi} \leftrightarrow \pi \in A_4)$.
6. For any $n \geqslant 13$ and good cycle $(a_1,\dots,a_n,b)$ and $w \notin \{a_1,\dots,a_n,b\}$ there is a successive triple (x,y,z) in the cycle such that $(Twxy \vee Twyx) \wedge (Twyz \vee Twzy) \wedge (Qwxyz \vee Qwyxz)$ holds.

Claim. *The class $\mathscr{C}$ of finite L-structures which satisfy Σ has the amalgamation property.*

Proof. Suppose that $B_1, B_2 \in \mathscr{C}$ have intersection A. We must specify structure on $D := B_1 \cup B_2$ so that D satisfies Σ. Using (b) above, we can define S on $B_1 \cup B_2$ so that any good cycle of D lies in B_1 or B_2. (List $B_1 \setminus A$ as $b_1, \dots, b_k$, and ensure that for any $i \in \{1, \dots, k\}$ and $x \in B_2 \setminus A$, Sb_ixy holds for all $y \in B_1 \cup B_2 \setminus (\{x, b_j : j \leqslant i\}$.) Thus, we only have to arrange that the last condition of Σ holds for any good cycle in B_1 and $w \in B_2 \setminus A$, and any good cycle in B_2 and $w \in B_1 \setminus A$. These are easy.

By the claim and Fraïssé's Theorem, there is a homogeneous countable model M. Let C be a finite subset of M. We show that M does not have non-algebraic strongly determined 1-types over C (from which the same result for n-types follows). To do this, we find a large finite set D of points in $M \setminus C$ such that $C \cup D$ is the domain of a good cycle $\sigma = (a_1, \dots, a_n, b)$ with $b \in D$. Using the homogeneity, we may choose D so that in addition the following hold.

(i) No successive pair of σ lies in C.

(ii) Any successive triple of σ contains at most one point of C.

(iii) if $\{x, y, z\} \subset C \cup D$ and $\{x, y, z\} \not\subseteq C$, then some ordering of $\{x, y, z\}$ satisfies S if and only if this is forced by part (a) of the definition of a good cycle.

(iv) Any two points of D have the same type (denoted p) over C.

(v) Any triple or quadruple from $C \cup D$ which satisfies T or Q contains at most one point from D.

(vi) There is no successive triple of σ which both meets C and contains the start of σ.

To see that (v) can be arranged, observe that by (vi) and (iii) any good cycle properly contained in $C \cup D$ lies entirely in C. Likewise, if $E \subset M \setminus C$, where $|E| \leqslant 6$ and every element of E realises p, then any good cycle in $C \cup E$ lies in C. For consider a counterexample σ'. Since σ' has at least 14 points, it contains a successive pair (y, z) from C. Its start b' must lie in E; for otherwise, there will be c, e on σ' with $c \in C$ and $e \in E$ and $Sb'ce$, and hence, if $d \in D \setminus \{b\}$, as $\mathrm{tp}(d/C) = \mathrm{tp}(e/C)$ we will have $Sb'cd$, contrary to (iii). But now, by (iii) again, $Sb'yz$ cannot hold, which is a contradiction as b' is the start of σ' and (y, z) is a successive pair.

Claim. (a) *If (u, w) is a successive pair of σ lying in $D \setminus C$ then (u, w), (w, u) have the same strong type over C.*

(b) *If (u, v, w) is a successive triple of σ with $u, w \in D \setminus C$ and $v \in C$, then (u, w), (w, u) have the same strong type over C.*

Proof of Claim. The same argument works for (a) and (b). Suppose that u, w are as in (a) or (b) and there is a finite equivalence relation $\sim$ over C and $u', w' \in M$ such that $(u, w) \sim (u', w')$. By (iii)–(vi), $\mathrm{tp}(uw/C) = \mathrm{tp}(wu/C)$. It follows from the remark before the claim that there are $u'', w'' \in M$ so that $\mathrm{tp}(u''w''u'w'/C) = \mathrm{tp}(u''w''wu/C) = \mathrm{tp}(uwu'w'/C)$. Then $(u'', w'') \sim (w, u)$ and $(u'', w'') \sim (u', w')$. Hence $(u, w) \sim (w, u)$, as required.

Now let $a \in M \setminus (C \cup D)$. Then there is a successive triple (u, v, w) from $C \cup D$ such that $(Tauv \vee Tavu) \wedge (Tavw \vee Tawv)$ holds. If say $u, v \in D$, then by the claim, (u, v) and (v, u) have the same strong type over C but $\mathrm{tp}(uva/C) \neq \mathrm{tp}(vua/C)$ (by (4)), so $\mathrm{tp}(a/C)$

does not extend to a strongly determined type over C. The same argument applies if $v, w \in D$. The remaining possibility is that $v \in C$ and $u, w \in D$. Now, however, the claim gives that (u,w) and (w,u) have the same strong type over C, but as some ordering of (a,u,v,w) satisfies Q, $\text{tp}(uw/aC) \neq \text{tp}(wu/aC)$ (by (5)). Thus, again, $\text{tp}(a/C)$ does not extend to a strongly determined type over C. □

We now sketch a second construction of an ω-categorical structure which does not admit strongly determined 1-types over any finite set. The result is weaker than Theorem 3.3, since the structure is not homogeneous in a finite relational language, but the construction technique is more flexible and rather different (though again based on circular orders).

Example 3.4. Choose a relational language $L=\{E_n, K_n\colon n \in \mathbf{N}, n>0\}$ such that for each n, E_n has arity $2n$ and K_n has arity $3n$. The structure M is built by a Fraïssé construction, so we first specify a class $\mathscr{C}$ of finite L-structures. In each structure $C \in \mathscr{C}$, each relation E_n determines an equivalence relation on the set (denoted $\binom{C}{n}$) of *unordered* n-element subsets of C, and K_n is interpreted by a circular order on the set $\binom{C}{n}/E_n$ of E_n-classes. It is easy to see that $\mathscr{C}$ is an amalgamation class, so there is a corresponding homogeneous structure M.

Suppose that $A := \{a_1, \ldots, a_{l-1}\}$ is a subset of M. We sketch a proof that no non-algebraic 1-type over A admits a strongly determined extension.

Let $Y := \{y_1, \ldots, y_l\}$, $Z := \{z_1, \ldots, z_l\}$, $Y' := \{y'_1, \ldots, y'_l\}$, $Z' := \{z'_1, \ldots, z'_l\}$ be disjoint l-sets (disjoint also from A), and put $\bar{y} := (y_1, \ldots, y_l)$, $\bar{z} := (z_1, \ldots, z_l)$, $\bar{y}' := (y'_1, \ldots, y'_l)$ and $\bar{z}' := (z'_1, \ldots, z'_l)$. Suppose that Y and Y' are E_l-equivalent, but inequivalent to Z, and Z and Z' are E_l-equivalent. Put $m := \binom{3l-1}{l} - 2$, and let $U_1, \ldots, U_m$ list the l-subsets of $\bar{a}\bar{y}\bar{z}$ apart from Y and Z, and $V_1, \ldots, V_m$ be the corresponding list for $\bar{a}\bar{z}'\bar{y}'$ (so the map $\bar{a}\bar{y}\bar{z} \mapsto \bar{a}\bar{z}'\bar{y}'$ takes each U_i to V_i). We may suppose that under C_l, the equivalence classes of the sets occur in the order $Y, U_1, \ldots, U_l, ZE_lZ', V_1, \ldots, V_l, Y'E_lY$. We may suppose also that for all $r \neq l$, all r-subsets of $A \cup Y \cup Z \cup Y' \cup Z'$ are E_r-equivalent. Clearly, such Y, Z, Y', Z' exist in M.

Let g be the partial map $\bar{a}\bar{y}\bar{z} \mapsto \bar{a}\bar{z}'\bar{y}'$. Then g is elementary, and it can be shown that g is strong over A (the details of this are tedious, and we omit them).

Now let p be any non-algebraic 1-type over A, and let $a_l \in M$ realise p. Put $A' := A \cup \{a_l\}$. The extension $\hat{g}$ of g fixing a_l does not extend to an automorphism, since it fixes the E_l-class of A' but swaps those of Y and Z, so does not preserve K_l. It follows that p does not have any strongly determined extension over A.

Finally, we show that a theory can admit strongly determined types over $\emptyset$ but not over some expansion by constants.

Example 3.5. Let X be an infinite set, $W = X \times \mathbf{Q}$, and consider the structure $M = (X, W)$ with a unary predicate for X, a projection map $\pi : W \to X$, and a ternary relation C interpreted as a circular order on each set $\pi^{-1}(x)$ $(x \in X)$. Then M is rich over $\emptyset$ and satisfies the joint embedding property with repsect to strong maps, so by Theorem 2.13, Th(M) admits strongly determined types over $\emptyset$. However, for any $a \in X$, the 1-type over a of $\pi^{-1}(a)$ carries a circular, order, so, as in Example 3.1, admits no strongly determined extension.

4. Multiplicity

In this section we attempt to control the different strongly determined extensions of a given type. We discuss a condition suggested by the Finite Equivalence Relation Theorem, consider the obvious notion of symmetry among strongly determined types, and investigate multiplicity. This leads to results on finite simple groups *involved* in a structure, motivated by finite axiomatisability considerations.

4.1. Finite equivalence relations

We first introduce an equivalence relation on the set of strongly determined types over a set of parameters. Two strongly determined types ρ and ρ' over A are said to have the same *direction* if there is an elementary permutation f of $\mathrm{acl}^{\mathrm{eq}}(A)$ fixing A pointwise such that for any M which is rich over A and $\bar{c} \models \rho_M$, $\bar{c}' \models \rho'_M$ (with $\bar{c}, \bar{c}' \in \mathbf{C}$), the following holds (where $\mathrm{acl}^{\mathrm{eq}}(A)$ is enumerated by the possibly infinite tupe $\bar{a}$): if $\bar{b}, \bar{b}' \in M$ and $\mathrm{tp}(f(\bar{a})\bar{b}') = \mathrm{tp}(\bar{a}\bar{b})$ then $\mathrm{tp}(\bar{c}'\bar{b}') = \mathrm{tp}(\bar{c}\bar{b})$. Essentially, this says that ρ and ρ' lie in the same orbit of $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(A)/A)$ in its natural action on strongly determined types over $\mathrm{acl}^{\mathrm{eq}}(A)$. In a stable theory, if $p \in S(A)$ then any two strongly determined extensions of p have the same direction.

Example 4.1. In the structure $(\mathbf{Q}, <)$, the unique 1-type over $\emptyset$ has two strongly determined extensions, corresponding to being greater than every element of $\mathbf{Q}$, or to being less than every element. These lie in different directions. In the reduct $(\mathbf{Q}, B)$, where B is the induced linear betweenness relation, the 1-type over $\emptyset$ again has two strongly determined extensions, but they lie in the same direction (apply any order-reversing permutation of $\mathbf{Q}$). For a very similar example, consider the random graph, viewed up to complementation.

We now discuss a Finite Equivalence Relation Theorem, which under certain conditions distinguishes between the types in a direction. Let ρ be a strongly determined type over A and $M \supset A$ be very rich over A. Let $M \subset B$. We say that a sequence $\bar{c}_1, \ldots, \bar{c}_n, \ldots$ is a *ρ-sequence over B* if $\bar{c}_{n+1} \models \rho_{B\bar{c}_1 \ldots \bar{c}_n}$, for every $n \in \omega$. It is clear that, given such a sequence, the type over M of any k-subsequence $\bar{c}_{i_1}, \ldots, \bar{c}_{i_k}$, $i_1 < \cdots < i_k$, is quasidefinable almost over A (this uses Lemma 1.3), and that the sequence is indiscernible over B. Moreover, this type is determined by k and ρ. Now we call a sequence $\bar{d}_1, \ldots, \bar{d}_n, \ldots$ a *ρ-sequence* if for all k, its k-subsequences realise the strong type over A just defined. We say that the Finite Equivalence Relation Theorem (FERT) holds in a direction Δ if for any distinct $\rho, \rho' \in \Delta$ the following holds: there is $k \in \omega$ and an

A-definable finite equivalence relation $E(\bar{x}_1,\dots,\bar{x}_k;\bar{y}_1,\dots,\bar{y}_k)$, such that if $\bar{c}_1,\dots,\bar{c}_k$ and $\bar{d}_1,\dots,\bar{d}_k$ are respectively ρ- and ρ'- k-sequences, then $\neg E(\bar{c}_1,\dots,\bar{c}_k;\bar{d}_1,\dots,\bar{d}_k)$. Also, $(\mathrm{FERT})_k$ holds, if FERT hold for some fixed positive integer k. The Finite Equivalence Relation Theorem for stable theories ensures that in the stable case, $(\mathrm{FERT})_1$ holds in every direction. The following lemma shows that FERT in a direction Δ is equivalent to the property that any strongly determined type $\rho \in \Delta$ is determined (among the elements of Δ) by its restriction over an infinite ρ-sequence.

Lemma 4.2. *Let Δ be a direction of strongly determined types over A. Then the following are equivalent.*

(i) *The* FERT *holds in the direction Δ.*

(ii) *For all $\rho,\rho' \in \Delta$ there is $k \in \omega$ such that if $\bar{c}_1 \dots \bar{c}_k$ is a ρ-sequence and $\bar{c} \models \rho_{A\bar{c}_1\dots\bar{c}_k}$ and $\bar{b} \models \rho'_{A\bar{c}_1\dots\bar{c}_k}$, then*

$$\mathrm{tp}(\bar{c}_1 \dots \bar{c}_k\bar{c}/A) \neq \mathrm{tp}(\bar{c}_1 \dots \bar{c}_k\bar{b}/A).$$

Proof. (i)$\Rightarrow$(ii): Suppose FERT holds in the direction Δ over A. Let $\rho,\rho' \in \Delta$ be distinct, and $C = (\bar{c}_1,\dots,\bar{c}_k,\dots)$ and $B = (\bar{b}_1,\dots,\bar{b}_k,\dots)$ be ρ- and ρ'-sequences respectively. Choose a corresponding finite equivalence relation $E(\bar{x}_1,\dots,\bar{x}_k;\bar{y}_1,\dots,\bar{y}_k)$, witnessing $(\mathrm{FERT})_k$ (for ρ,ρ'), with k as small as possible. We may assume that $2 \leqslant k$ (for if $k=1$ then $\rho_{A\bar{c}_1}$ and $\rho'_{A\bar{c}_1}$ are different, and (ii) follows). Thus the sequences $\bar{c}_1,\dots,\bar{c}_{k-1}$ and $\bar{b}_1,\dots,\bar{b}_{k-1}$ realise the same type over $\mathrm{acl}^{\mathrm{eq}}(A)$. So, suppose that M is a sufficiently saturated model containing A and all the parameters in B and C, and $\bar{b} \models \rho'_M$. Then by Lemma 1.3, $\bar{c}_1,\dots,\bar{c}_{k-1},\bar{b}$ and $\bar{b}_1,\dots,\bar{b}_{k-1},\bar{b}$ realise the same type over $\mathrm{acl}^{\mathrm{eq}}(A)$. It follows that the E-classes of $\bar{c}_1,\dots,\bar{c}_k$ and $\bar{c}_1,\dots,\bar{c}_{k-1},\bar{b}$ are distinct. Hence, if $\bar{c} \models \rho_{A\bar{c}_1\dots\bar{c}_k}$ then $\bar{c}$ and $\bar{b}$ have different types over $A\bar{c}_1\dots\bar{c}_k$.

(ii)$\Rightarrow$(i): Again, suppose that $\rho,\rho' \in \Delta$ are distinct, and choose $C,B,\bar{b}$ as in the implication (i)$\Rightarrow$(ii), and $\bar{c} \models \rho_{A\bar{c}_1\dots\bar{c}_k}$. Then

$$\mathrm{tp}(\bar{c}_1 \dots \bar{c}_k\bar{c}/A) \neq \mathrm{tp}(\bar{c}_1 \dots \bar{c}_k\bar{b}/A),$$

but as ρ,ρ' are in the same direction,

$$\mathrm{tp}(\bar{c}_1 \dots \bar{c}_k\bar{c}/A) = \mathrm{tp}(\bar{b}_1 \dots \bar{b}_k\bar{b}/A).$$

It follows that the map $(\bar{c}_1 \dots \bar{c}_k) \mapsto (\bar{b}_1 \dots \bar{b}_k)$ is not elementary over $\bar{b}A$. By the definition of strongly determined types this map is not elementary over $\mathrm{acl}^{\mathrm{eq}}(A)$, and the existence of the required finite equivalence relation follows. $\square$

We now give two examples, both ω-categorical. The first shows a direction without FERT, whilst the second exhibits a direction which satisfies FERT but does not satisfy $(\mathrm{FERT})_k$ for any $k \in \omega$.

Example 4.3. Let M be a structure in a language of a unary predicate P and two binary relations E and R. Let P be infinite and coinfinite in M, E define an equivalence relation on P with two infinite classes and $R \subset P \times (M \backslash P)$ be a symmetric relation, which is random with respect to E: for any finite disjoint $A, B \subset P$ there exists $c \in M \backslash P$ having no edges with A and adjacent to any element of B. This structure is homogeneous. Choose two distinct strongly determined types ρ and ρ' of $\neg P$, such that $\rho_M(x)$ says that x is R-adjacent to all members in M of one E-class, and none in the other, and ρ' is the same but with the E-classes reversed. The types ρ and ρ' are in the same direction (consider an automorphism interchanging the E-classes). Since $\mathrm{Aut}(M)$ is k-transitive on $M \backslash P$ for every positive integer k, ρ and ρ' cannot be distinguished by any finite equivalence relation on the set of tuples of $M \backslash P$.

In Example 4.3, if $c_1, \ldots, c_n$ is a ρ-sequence and $d_1, \ldots, d_n$ is a ρ'-sequence then the map $\bar{c} \to \bar{d}$ can be extended to an elementary map fixing some model. This shows that it is possible that sequences corresponding to distinct strongly determined types of the same direction have the same *Lascar strong types* (see [29]).

Example 4.4. This is the example of Cherlin and Hrushovski described in [15]. Let M be a structure of the language $(E^n: n \in \omega)$ where every E^n is interpreted as an equivalence relation on the set of all n-tuples having pairwise distinct elements. Moreover, we demand that every E^i has exactly two classes P_i^+ and P_i^- and the expansion of M by all P_i^+, P_i^- is universal homogeneous.

Let $p(x) \in S(M)$ be a type asserting that for every positive integer n, every n-tuple $\bar{a} \in M$ of distinct elements the tuple $\bar{a}x$ is in P_{n+1}^+. It is easily seen that $p(x)$ is definable almost over $\emptyset$. Let Δ be the direction of $p(x)$ and $m \in \omega$. Let $p_m(x) \in S(M)$ be the type asserting that for any $\bar{a} \in M$ as above, $\bar{a}x \in P_{n+1}^+$ if $n < m$ and $\bar{a}x \in P_{n+1}^-$ otherwise. It is clear that Δ contains the strongly determined type over $\emptyset$ which corresponds to p_m. Sequences corresponding to p and p_m respectively have the same strong type over $\emptyset$ if and only if their length is at most m. This shows that any $(\mathrm{FERT})_k$ does not hold in Δ. On the other hand it is obvious that FERT holds in Δ.

An obvious example where $\neg(\mathrm{FERT})_1 \wedge (\mathrm{FERT})_2$ holds over $\emptyset$ is the countable model of the dense linear betweenness relation. We do not know general conditions under which FERT implies $(\mathrm{FERT})_k$ for some fixed k, but can prove this in the symmetric case (which includes the stable case) described next.

4.2. *Symmetric sets of types*

We say that a set Δ of strongly determined types over A is *symmetric* if the following holds: whenever $M \supset A$ is very rich over A, and $\rho, \rho' \in \Delta$ (not necessarily distinct), and $\bar{c} \models \rho_M$, $\bar{b} \models \rho'_{M\bar{c}}$, the type of $\bar{c}$ over $M\bar{b}$ is equal to $\rho_{M\bar{b}}$. By forking symmetry, if the ambient theory is stable and Δ is any set of types which do not fork over A, then Δ is a symmetric set. For the random graph, if ρ is either of the two strongly determined 1-types over $\emptyset$, then $\{\rho\}$ is symmetric. However, for the random graph there are non-symmetric strongly determined 2-types: let M be the random graph and $\rho_M(x,y)$ assert that x is joined to all members of M, and y to no members of M.

Proposition 4.5. *Suppose that $M \supset A$ is very rich over A and let Δ be a symmetric set of strongly determined types over A.*
(i) *If $\rho \in \Delta$ then any ρ-sequence over M is an indiscernible set over M.*
(ii) *If Δ is a direction over A and satisfies* FERT, *then it satisfies* $(\mathrm{FERT})_1$.

Proof. We start with the following claim.

Claim. *For any $\rho, \rho' \in \Delta$, any ρ-sequence $\bar{c}_1, \ldots, \bar{c}_k$ over M, and any $\bar{b} \models \rho'_{M\bar{c}_1\ldots\bar{c}_k}$, the sequence $\bar{c}_1 \ldots \bar{c}_k$ is a ρ-sequence over $M\bar{b}$.*

The proof is by induction (so the inductive hypothesis is that the claim holds for k, for *any* $A, \Delta, M, \rho, \rho'$). For $k = 1$ this is the above definition of a symmetric set. In order to prove the claim for $k+1$, choose a ρ-sequence over M, say $\bar{c}_1, \ldots, \bar{c}_{k+1} \models \rho_M$. Also choose tuples $\bar{b} \models \rho'_{M\bar{c}_1\ldots\bar{c}_{k+1}}$ and $\bar{b}' \models \rho'_{M\bar{c}_1}$ such that the sequence $\bar{c}_2 \ldots \bar{c}_{k+1}$ is a ρ-sequence over $M\bar{c}_1\bar{b}'$.

Notice that the sequence $\bar{c}_1 \ldots \bar{c}_{k+1}$ is a ρ-sequence over $M\bar{b}'$. This follows because $\bar{c}_2, \ldots, \bar{c}_{k+1}$ is a ρ-sequence over $M\bar{b}'\bar{c}_1$, and by the case $k=1$ we have $\bar{c}_1 \models \rho_{M\bar{b}'}$.

To prove the claim for $k+1$, we suppose that $\bar{m} \in M$, and prove

$$\mathrm{tp}(\bar{m}\bar{b}\bar{c}_1 \ldots \bar{c}_{k+1}/A) = \mathrm{tp}(\bar{m}\bar{b}'\bar{c}_1 \ldots \bar{c}_{k+1}/A).$$

Choose $\bar{c}_1' \in M$ such that the tuples $\bar{m}\bar{c}_1$ and $\bar{m}\bar{c}_1'$ have the same strong type over A. Then,

$$\mathrm{tp}(\bar{m}\bar{b}\bar{c}_1 \ldots \bar{c}_{k+1}/A) = \mathrm{tp}(\bar{m}\bar{b}\bar{c}_1' \ldots \bar{c}_{k+1}/A).$$

(For by Lemma 1.3, $\mathrm{tp}(\bar{c}_2 \ldots \bar{c}_{k+1}/M\bar{c}_1)$ is quasidefinable almost over A, so $\bar{m}\bar{c}_1\bar{c}_2 \ldots \bar{c}_{k+1}$ and $\bar{m}\bar{c}_1'\bar{c}_2 \ldots \bar{c}_{k+1}$ realise the same strong types over A, and the assertion follows from the definition of strongly determined type.) Similarly,

$$\mathrm{tp}(\bar{m}\bar{b}'\bar{c}_1 \ldots \bar{c}_{k+1}/A) = \mathrm{tp}(\bar{m}\bar{b}'\bar{c}_1' \ldots \bar{c}_{k+1}/A).$$

Also, by the assumption that the claim holds for k, $\bar{c}_2, \ldots, \bar{c}_{k+1}$ is a ρ-sequence over $M\bar{b}$, so

$$\mathrm{tp}(\bar{m}\bar{b}'\bar{c}_1'\bar{c}_2 \ldots \bar{c}_{k+1}/A) = \mathrm{tp}(\bar{m}\bar{b}\bar{c}_1'\bar{c}_2 \ldots \bar{c}_{k+1}/A).$$

Hence,

$$\mathrm{tp}(\bar{m}\bar{b}\bar{c}_1 \ldots \bar{c}_{k+1}/A) = \mathrm{tp}(\bar{m}\bar{b}'\bar{c}_1 \ldots \bar{c}_{k+1}/A),$$

as required.

(i) Let $\bar{c}_1, \bar{c}_2, \ldots$ be a ρ-sequence over M. Since all ρ-sequences over M of the same length are isomorphic over M, for each i there is M_i containing $M\bar{c}_1 \ldots \bar{c}_i$ such that $\bar{c}_{i+1},\bar{c}_{i+2},\ldots$ is a ρ-sequence over M_i. It follows from the claim that for each i, $\bar{c}_i,\bar{c}_1,\ldots,\bar{c}_{i-1}$, is a ρ-sequence over M, and hence $\bar{c}_i,\bar{c}_1,\bar{c}_2,\ldots$ is a ρ-sequence over M. Part (i) follows easily.

(ii) Let $\rho,\rho'\in\Delta$ be distinct. By Lemma 4.2, for some k we can choose $\bar{c}_1,\ldots,\bar{c}_k$ and $\bar{b}$ as in the claim such that additionally the types of $\bar{c}_k$ and $\bar{b}$ over $\bar{c}_1\ldots\bar{c}_{k-1}A$ are different. We must show that $\bar{c}_1$ and $\bar{b}$ realise different strong types over A. By the claim and (i),

$$\mathrm{tp}(\bar{c}_2\ldots\bar{c}_k\bar{b}/A)=\mathrm{tp}(\bar{c}_1\ldots\bar{c}_{k-1}\bar{b}/A)\neq\mathrm{tp}(\bar{c}_1\ldots\bar{c}_k/A)=\mathrm{tp}(\bar{c}_2\ldots\bar{c}_{k-1}\bar{c}_1/A),$$

so the map $\bar{c}_1\to\bar{b}$ is not elementary over $\bar{c}_2\ldots\bar{c}_kA$. Hence it suffices to show that $\bar{c}_2\ldots\bar{c}_k$ is a ρ-sequence over $M\bar{b}\bar{c}_1$ (then we can apply the definition of a strongly determined type over A). To see this, choose $\bar{d}_2\ldots\bar{d}_k$, a ρ-sequence over $M\bar{b}\bar{c}_1$. Then $\bar{c}_1\bar{d}_2\ldots\bar{d}_k$ is a ρ-sequence over $M\bar{b}$ (because Δ is symmetric, so $\bar{c}_1\models\rho_{M\bar{b}}$). By the claim $\bar{c}_1,\ldots,\bar{c}_k$ is a ρ-sequence over $M\bar{b}$. So,

$$\mathrm{tp}(\bar{c}_1\ldots\bar{c}_k/M\bar{b})=\mathrm{tp}(\bar{c}_1\bar{d}_2\ldots\bar{d}_k/M\bar{b}).$$

Hence

$$\mathrm{tp}(\bar{c}_2\ldots\bar{c}_k/M\bar{b}\bar{c}_1)=\mathrm{tp}(\bar{d}_2\ldots\bar{d}_k/M\bar{b}\bar{c}_1),$$

as required. □

We remark that if M is the smoothly approximated structure consisting of a vector space over a finite field endowed with a symplectic form, then every set of strongly determined types over every finite set is symmetric, even though M is unstable (though it is supersimple). This is essentially because if $\rho(\bar{x})$ is such a type (assumed say to be strongly determined over $\emptyset$) and $\bar{c}\models\rho_M$, then every member of $\bar{c}$ is orthogonal to M with respect to the symplectic form.

Problem 4.6. Let T be a complete theory admitting strongly determined types such that for every set A, the set of all strongly determined types over A is symmetric. Must T be without the strict order property?

The following argument is slight evidence for a positive answer. Let T be a complete theory admitting strongly determined types, and suppose that the formula $\phi(\bar{x},\bar{y})$ witnesses the strict order property for T: that is, it defines a partial order on M^n with an infinite chain $\{\bar{a}_k\colon k\in I\}$. By compactness, we may assume that the $\bar{a}_k$ form an indiscernible $\mathbf{Z}$-sequence of the same strong type (denoted by p). Now if ρ is a strongly determined $\bar{x}$-type over $\emptyset$ that extends p, then each of the conditions $\phi(\bar{x},\bar{a}_k)\in\rho, k\in\mathbf{Z}$ and $\phi(\bar{a}_k,\bar{x})\in\rho, k\in\mathbf{Z}$ implies that $\{\rho\}$ is not symmetric. Indeed, suppose that $\phi(\bar{x},\bar{a}_k)\in\rho$, where $k\in\mathbf{Z}$. Then for $\bar{b}_1\models\rho_M$ and $\bar{b}_2\models\rho_{M,\bar{b}_1}$ we have $\phi(\bar{b}_2,\bar{b}_1)\wedge\neg\phi(\bar{b}_1,\bar{b}_2)$, so $\bar{b}_1\not\models\rho_{M,\bar{b}_2}$.

We do not have an example of a non-simple theory admitting strongly determined types, in which the set of all strongly determined types over any parameter set is symmetric. In the universal homogeneous triangle-free graph Γ (whose theory is not simple but does not have the strict order property), the unique strongly determined type over $\emptyset$ is symmetric. However, let a be a vertex, and $\Gamma(a)$ and $\Delta(a)$ be the sets of neighbours and non-neighbours respectively, and let $p(x)$ be the 1-type over a of non-neighbours of a, and q the 1-type of neighbours of a. Now p has a strongly determined extension ρ^p such that ρ^p_Γ is realised by vertices joined to all of $\Gamma(a)$ and none of $\Delta(a)$, and q has a strongly determined extension ρ^q such that ρ^q_Γ is realised by vertices joined to none of $\Gamma(a) \cup \Delta(a)$. Then $\{\rho^p, \rho^q\}$ is not symmetric. (However, ρ^p, ρ^q do not extend the *same* type over a.)

4.3. Multiplicity and involved finite groups

Throughout this subsection we shall assume that M is very rich over $\emptyset$. Let $\bar{a} \in M$ and $D_{\bar{a}}$ be the set of all types from $S(M)$ which are quasidefinable almost over $\bar{a}$. For any $A \subset M$ containing $\bar{a}$, let H_A denote the group of elementary partial maps on $\mathrm{acl}^{\mathrm{eq}}(A)$ over A. Then H_A acts on $D_{\bar{a}}$ (as described at the beginning of Section 4.1) and hence on $D_{\bar{a}} \times \mathrm{acl}^{\mathrm{eq}}(A)$. If $\bar{c} \in \mathbf{C}^{\mathrm{eq}}$ and $p(\bar{x}) = \mathrm{tp}(\bar{c}/A)$ has an extension to a type in $D_{\bar{a}}$, define the *multiplicity* of p with respect to $\bar{a}$ to be the greatest size of an H_A-orbit on $D_{\bar{a}}$ containing an extension of p. This multiplicity is denoted by $\mathrm{mult}_{\bar{a}}(\bar{c}/A)$, or $\mathrm{mult}_{\bar{a}}(p)$, or by $\mathrm{mult}(\bar{c}/A)$ if $\bar{a}$ enumerates A. (In this last case, $\bar{a}$ can be infinite, but we then need M to be very rich over A.) Observe that if $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$, then $\mathrm{mult}(\bar{b}/A)$ is just the number of translates of $\bar{b}$ over A.

The following lemma is now a straightforward generalisation of a well-known fact for ω-stable theories.

Lemma 4.7. *Let $\bar{a} \in A \subset M$ and suppose that M is very rich over A. Let $p(\bar{x}) \in S(A)$ extend to a type from $S(M)$ quasidefinable almost over $\bar{a}$ and suppose that the multiplicity $\mathrm{mult}_{\bar{a}}(p)$ is finite. Then for any $\bar{c} \in M$ realising $p(\bar{x})$ and any $\bar{\mathbf{b}} \in \mathrm{acl}^{\mathrm{eq}}(A)$*

$$\mathrm{mult}(\bar{\mathbf{b}}/A) \leqslant \mathrm{mult}_{\bar{a}}(\bar{c}/A) \cdot \mathrm{mult}(\bar{\mathbf{b}}/A\bar{c}).$$

Proof. Let B be the H_A-orbit of $\bar{\mathbf{b}}$. There is an A-definable finite equivalence relation E and $\bar{b} \in M$ such that $\bar{\mathbf{b}} = \bar{b}/E$. As $p(\bar{x})$ extends to a type quasidefinable almost over $\bar{a}$, by richness we can choose $\bar{b}$ such that $\mathrm{tp}(\bar{c}/A\bar{b})$ extends to a type q which is still quasidefinable almost over $\bar{a}$. Let $\bar{c}'$ realise q, let $Q \subset S(M)$ be the H_A-orbit of q, and let P be the H_A-orbit of $(\bar{\mathbf{b}}, q)$ in $B \times Q$. Now $\mathrm{mult}(\bar{\mathbf{b}}/A) = |B| \leqslant |P|$ and $|P| \leqslant \mathrm{mult}_{\bar{a}}(\bar{c}/A) \cdot \mathrm{mult}(\bar{\mathbf{b}}/A\bar{c}')$. Since $\bar{c}$ and $\bar{c}'$ have the same types over $A\bar{b}$, we obtain the required inequality. $\square$

With Theorem 4.11 in mind, we now define the (very rich over $\emptyset$) structure M to be an *FM-structure* if $\mathrm{Th}(M)$ admits strongly determined types and, for each finite $A \subset M$ and $\bar{c} \in M$, $\mathrm{mult}(\bar{c}/A)$ is bounded by a natural number only depending on $|A|$ and the length of $\bar{c}$. This is a property of $\mathrm{Th}(M)$, so we also talk of *FM-theories*.

Observe that algebraically closed fields are not FM, even though multiplicity is finite in any ω-stable structure.

Recall that an ω-categorical structure M is *G-finite* [34] if, for every finite $A \subset M$, $|\mathrm{Aut}(M/A) : \mathrm{Aut}^o(M/A)|$ is finite.

Lemma 4.8. *Suppose that M is ω-categorical and admits strongly determined types over any finite set, and that one of the following conditions holds.*
(i) *M is homogeneous in a finite relational language of arity l;*
(ii) *M is G-finite;*
(iii) *every direction over every finite set satisfies* $(\mathrm{FERT})_k$ *for some $k \in \omega$ (depending on the direction);*
(iv) *M is stable.*
Then M is FM.

Proof. Let $\bar{a}$ enumerate the finite set A.

(i) The action of H_A on $D_{\bar{a}}$ is determined by its action on the set of classes of A-definable finite equivalence relations on k-tuples, where $k \leqslant l-1$. By ω-categoricity, this bounds the length of each H_A-orbit on $D_{\bar{a}}$.

(ii) In this case $\mathrm{mult}(\bar{c}/A) \leqslant |\mathrm{Aut}(M/A)\colon \mathrm{Aut}^o(M/A)|$, which is finite.

(iii) Any H_A-orbit on $D_{\bar{a}}$ lies within a direction. Since any two types in a direction are distinguished by a finite equivalence relation on k-tuples (for the corresponding k), and by ω-categoricity there are finitely many finite equivalence relations on k-tuples over A, the result follows.

(iv) This follows by the usual (stable) Finite Equivalence Relation Theorem, and the argument in (iii). □

Observe that in (i) and (ii), $\mathrm{mult}(\bar{c}/A)$ is bounded by a natural number depending only on $|A|$. Example 4.4 shows that it is not true that every ω-categorical structure which admits strongly determined types and satisfies FERT is FM.

In [25], in a proof that any finitely axiomatisable uncountably categorical theory is locally modular, Hrushovski introduces the following notion: a finite simple group S is *involved* in a theory T if there exist $A \subset M \models T$, a finite $B_0 \subset M^{\mathrm{eq}}$ and $B \subset \mathrm{dcl}^{\mathrm{eq}}(A \cup B_0) \cap \mathrm{acl}^{\mathrm{eq}}(A)$ such that the following hold:
1. B is invariant under all elementary maps over A,
2. the group H_A defined before Lemma 4.7 induces a finite permutation group H_A^B on B,
3. S is a composition factor of H_A^B.

It is proved in [25] that a finitely axiomatisable ω_1-categorical theory cannot involve infinitely many finite simple groups. The main tool in [25] is the notion of Morley multiplicity of types. Now we will show that the method of Hrushovski is applicable in other settings with strongly determined types.

If $\bar{a} \in M$, we define the *width* of a formula $\phi(\bar{x}, \bar{a})$ (denoted $\mathrm{wid}(\phi(\bar{x}, \bar{a}))$) to be the minimal multiplicity of a complete type over $\bar{a}$ which contains $\phi(x, \bar{a})$ and has a strongly determined extension over $\bar{a}$, and to be ∞ if there is no such complete type. Then for a set Σ of $\forall\exists$-sentences true of M we define μ_Σ to be

$$\mathrm{Max}\{\mathrm{wid}(\phi(\bar{x},\bar{b})):\ \phi(\bar{x},\bar{y}) \text{ is quantifier-free, } \bar{b}\in M, \forall\bar{y}\exists\bar{x}\phi(\bar{x},\bar{y})\in\Sigma)\},$$

if this is finite, and to be ∞ otherwise. Notice that in an ω-stable theory any formula has finite width. On the other hand, if an ω-categorical theory has the property that every formula has finite width, then the theory is FM.

The proof of the following proposition is almost the same as the corresponding one in [25].

Proposition 4.9. *Let T be a countable model-complete theory admitting strongly determined types and axiomatised by a set Σ of $\forall\exists$-sentences such that μ_Σ is finite. Then any finite simple group involved in T has size at most $\mu_\Sigma!$.*

Proof. Suppose that $M\models T$, $A\subset M$ with $|A|<|M|$, and that $B\subset \mathrm{acl}^{\mathrm{eq}}(A)$ satisfies the conditions from the definition of *involved*. We may suppose that M is κ^+-saturated, where $\kappa:=\mathrm{Max}\{|A|,\aleph_0\}$. Construct a structure $N\models T$ as follows. Let $A_0=A$ and at limit stages λ, $A_\lambda=\bigcup_{\alpha<\lambda}A_\alpha$. At stage $\nu+1$ the set $A_{\nu+1}$ is obtained by adding a realisation $\bar{c}\in M$ of some $\phi(\bar{x},\bar{b})$, where $\bar{b}\subset A_\nu$ and $\forall\bar{y}\exists\bar{x}\phi(\bar{x},\bar{y})\in\Sigma$, and $\bar{c}$ is chosen so that $\mathrm{tp}(\bar{c}/A_\nu)$ extends to a strongly determined type over $\bar{b}$ with $\mathrm{mult}_{\bar{b}}(\bar{c}/A_\nu)\leqslant\mu_\Sigma$. Let $N=\bigcup(A_\nu:\ \nu<\kappa)$. Since Σ axiomatises T, we can arrange that $N\models T$. Hence, by model-completeness of T, N is an elementary substructure of M. So in particular, $B\subset N^{\mathrm{eq}}$.

Let H_α denote the group of permutations of B induced by elementary maps which fix A_α pointwise and have domain and range containing M. Using Lemma 4.7, we see that $H_{\alpha+1}$ has index at most μ_Σ in H_α. An easy combinatorial argument (for example, see [25]) shows that every composition factor of H_0 has size at most $\mu_\Sigma!$. $\square$

Part of our motivation here is a conjecture from [37].

Conjecture 4.10. *If T is a finitely axiomatisable ω-categorical theory then T has the strict order property.*

The conjecture has been confirmed in some partial cases. It was proved when algebraic closure is trivial (that is, $\mathrm{acl}(A)=A$ for all A) in [37], and under the more general condition of distributive algebraic closure in [26], and in the ω-stable case it follows from Corollary 7.4 of [5]. The conjecture, and the results in [25], raise delicate questions about algebraic closure in finitely axiomatised theories.

Suppose that T is an ω-categorical FM-theory. Let Σ be a finite set of axioms for T, and let Δ be the set of all subformulas of Σ. Considering Δ as a set of new relational symbols we get an expansion M' of $M\models T$ axiomatised by a finite set Σ' of $\forall\exists$-axioms. Clearly, $\mathrm{Th}(M')$ is a model complete, ω-categorical and finitely axiomatisable FM-theory. Thus, in this situation we may suppose that T is model-complete, and that the axioms in Σ are $\forall\exists$. Clearly, μ_Σ is finite. The following theorem now links the material of this section.

Theorem 4.11. *Let T be an ω-categorical theory, axiomatised by a finite set Σ of $\forall\exists$-sentences, and suppose that T admits strongly determined types and satisfies one of the conditions of Lemma* 4.8. *Then T is an FM-theory and hence any finite simple group involved in T has order at most $\mu_\Sigma!$.*

It is not known whether there is any finitely axiomatisable stable ω-categorical theory. However, the above theorem has the following corollary for such a structure. We need the following definition from [45]: a finite set F is *coded* in a theory T if there is a tuple $\bar{b}$ such that an automorphism σ fixes F as a set if and only if $\sigma(\bar{b})=\bar{b}$.

Corollary 4.12. *Let T be a stable ω-categorical theory axiomatised by a finite set Σ of $\forall\exists$-sentences. Let $n\in\omega$ with $n\geqslant\mu_{\Sigma!}$, and suppose that there is a simple group S with $|S|=n$. Then there is an n-element set F of pairs which is not coded in T.*

Proof. If T is a counterexample then applying the proof of Theorem 3.9 from [45] we obtain a finite A and an algebraic $q(x)\in S(A)$ such that for any $a\models q(x)$ the set $B=\mathrm{dcl}(Aa)$ is invariant under $\mathrm{Aut}(M/A)$ and $\mathrm{Aut}(B/A)$ is isomorphic to S. This contradicts Theorem 4.11. □

We now give another application of Proposition 4.9.

Corollary 4.13. *Let T be a countable weakly o-minimal theory. Then any group involved in T is trivial.*

Proof. Extending the language by definable relations we expand T to a model-complete theory T^*. Since T^* is weakly o-minimal too, we may apply Theorem 2.6. The proof of this theorem shows that each 1-type has multiplicity 1. The language of the theory T^* can be chosen so that T^* is axiomatised by a set Σ of sentences of the form $\forall\bar{y}\exists x\phi(x,\bar{y})$ (see [37]). Applying Proposition 4.9 we obtain the result. □

5. Covers of ω-categorical structures

In this section we show that a smoothly approximated structure has a (non-algebraic) strongly determined type over some one-element set. The main ingredients of this result are Lie coordinatisability of smoothly approximated structures [6] and Lemma 5.2 below. The latter guarantees (under a G-finiteness assumption) that the process of taking finite covers preserves existence of non-algebraic strongly determined types. We apply these ideas in Theorem 5.5, and obtain a description of superlinked finite covers of structures which admit strongly determined types and have certain other nice properties (Theorem 5.6).

We start this section with an existence result (for strongly determined types) in a quite general setting. Suppose that L, L^+ are first-order languages and $P \in L^+ \setminus L$ is a unary predicate, M is an L^+-structure, and that P^M is the domain of a substructure N of the reduct $M|L$. Following Hodges and Pillay [24], we say that M is a *symmetric extension* of N if the restriction mapping $\mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is surjective. We adopt the notation S_M, S_N, tp_M, tp_N, to indicate whether types are considered with respect to $\mathrm{Th}(M)$ (in L^+) or $\mathrm{Th}(N)$ (in L) (so $S_M(N)$ denotes the set of types over the set N of $\mathrm{Th}(M)$).

Proposition 5.1. *Let M be an ω-categorical symmetric extension of some $\emptyset$-definable substructure N. Let $p(\bar{x}) \in S_N(N)$ be a type definable almost over $\bar{b} \in N$. Then there is a type $q(\bar{x}) \in S_M(M)$ which is definable almost over $\bar{b}$ and implies $p(\bar{x})$.*

Proof. We may suppose that $\bar{b} = \emptyset$. Let $\phi(\bar{x}, \bar{z})$ be a $\mathrm{Th}(M)$-complete L^+-formula which implies $\bar{x} \in N$, and let $r(\bar{z})$ be the $\mathrm{Th}(M)$-complete type determined by this formula. By Lemma 7(b) from [24] there exists a $\mathrm{Th}(M)$-complete type $s(\bar{y}, \bar{z})$ implying $\bar{y} \in N \wedge r(\bar{z})$, such that for any $\bar{a}\bar{a}' \models s$ the type $\mathrm{tp}_M(\bar{a}'/N)$ is definable over $\bar{a}$. Hence there is a formula $\delta(\bar{x}, \bar{y})$ such that for any $\bar{a}\bar{a}' \models s$, $\delta(N, \bar{a}) = \phi(N, \bar{a}')$, and as M is a symmetric extension of N, we may suppose that δ is an L-formula. Let $\psi(\bar{y})$ be an L-formula over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ such that $\psi(N) = \{\bar{a} : \delta(\bar{x}, \bar{a}) \in p(\bar{x})\}$. It is easily seen that the L^+-formula $\exists \bar{y}(s(\bar{y}, \bar{z}) \wedge \psi(\bar{y}))$ is over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. We put $\phi(\bar{x}, \bar{c})$ into $q(\bar{x})$ if and only if $\bar{c}$ realizes this formula.

We claim that $q(\bar{x})$ has the required properties. First, to see that $p \subseteq q$, let $\phi(\bar{x}, \bar{c}) \in p$. Then $\bar{c} \in N$, $s(\bar{y}, \bar{z})$ can be taken as the formula $\bar{y} = \bar{z}$, and δ can be taken to be ϕ. Then $\psi(\bar{c})$ holds, so $\phi(\bar{x}, \bar{c}) \in q$. To see that q is consistent, suppose that $\phi_i(\bar{x}, \bar{c}_i) \in q$, with corresponding formulas ψ_i, δ_i and complete types r_i, s_i (for $i = 1, \ldots, k$). Then

$$\bigwedge_{i=1}^{k} \exists \bar{y}_i (s_i(\bar{y}_i, \bar{c}_i) \wedge \psi_i(\bar{y}_i)),$$

so there are $\bar{a}_1, \ldots, \bar{a}_k \in N$ such that

$$\bigwedge_{i=1}^{k} (s_i(\bar{a}_i, \bar{c}_i) \wedge \psi_i(\bar{a}_i)).$$

Hence $\delta_i(\bar{x}, \bar{a}_i) \in p$ for $i = 1, \ldots, k$. Choose $\bar{b} \in N$ realising the restriction of p over $\bar{a}_1 \ldots \bar{a}_k$. Then $\bigwedge_{i=1}^{k} \delta_i(\bar{b}, \bar{a}_i)$, so $\bigwedge_{i=1}^{k} \phi_i(\bar{b}, \bar{c}_i)$, so $\{\phi_i(\bar{x}, \bar{c}_i)\colon 1 \leqslant i \leqslant k\}$ is consistent. The fact that q is definable almost over $\emptyset$ is trivial. $\square$

We now generalise the situation. Let M be an ω-categorical structure in a language L_0 and let M^+ be an ω-categorical extension of M by definitions. This means that the language L^+ of M^+ is an extension of L_0 by finitely many eq-sorts and eq-relations 0-definable in M. Note that $\mathrm{Aut}(M) = \mathrm{Aut}(M^+)$ and $\mathrm{Aut}^o(M) = \mathrm{Aut}^o(M^+)$ (as topological groups). Let $L \subset L^+$ and M^+ be a symmetric extension of an L-structure N (whose domain is the interpretation of $P \in L^+$). Let $\bar{d} \in N$ and $\bar{c} \in M$ satisfy $\mathrm{acl}_{L^+}(\bar{c}) = \mathrm{acl}_{L^+}(\bar{d})$. Assume that $p(\bar{x})$ is a type of the structure $(M, \bar{c})$ and $q_p(\bar{y})$ is a type of the structure $(N, \bar{d})$ such that for any $\bar{a} \models q_p$ there is $\bar{b} \models p$ with $\bar{b} \in \mathrm{acl}_{L^+}(\bar{a}\bar{d})$. Under these hypotheses, we have

Lemma 5.2. *Let M be G-finite. If $q_p(\bar{y})$ extends to a strongly determined L-type ρ over $\bar{d}$ (with respect to* Th(N)*), then $p(\bar{x})$ extends to a strongly determined type (with respect to* Th(M)*) over $\bar{c}$.*

Proof. We work in an ω_1-saturated elementary extension of M^+. By Proposition 5.1 we can view ρ as a strongly determined type over $\bar{d}$ with respect to Th(M^+) (over L^+). Choose $\bar{a} \models \rho_{M^+}$ and $\bar{b} \in \mathrm{acl}_{L^+}(\bar{a}\bar{d})$ such that $\bar{b} \models p(\bar{x})$. Define

$$H := \{g \in \mathrm{Aut}(M/\bar{c}) \colon g \cup \mathrm{id}_{\bar{b}} \text{ is } \mathrm{Th}(M^+, \bar{c})\text{-elementary}\},$$

$$F := \{g \in \mathrm{Aut}(M^+/\bar{d}) \colon g \cup \mathrm{id}_{\bar{a}} \text{ is } \mathrm{Th}(M^+, \bar{d})\text{-elementary}\}.$$

Note that as elementary maps are finitely determined, H is a closed subgroup of Aut(M) and F is a closed subgroup of Aut(M^+) (and hence of Aut(M)).

Since $\bar{d}$ and $\bar{c}$ are interalgebraic, the group Aut$(M^+/\bar{c}\bar{d})$ is of finite index in both Aut$(M^+/\bar{d})$ and Aut$(M/\bar{c})$. By the choice of $\bar{a}$, we have $\mathrm{Aut}^o(M^+, \bar{d}) \leqslant F$ and, by G-finiteness, $|\mathrm{Aut}(M^+/\bar{d})\colon \mathrm{Aut}^o(M^+/\bar{d})|$ is finite, so F is of finite index in Aut$(M^+, \bar{d})$. It follows that $F \cap \mathrm{Aut}(M/\bar{c})$ is of finite index in Aut$(M/\bar{c})$. Since $\bar{b}$ is algebraic over $\bar{a}\bar{d}$, the group $H \cap F \cap \mathrm{Aut}(M/\bar{c})$ is of finite index in $F \cap \mathrm{Aut}(M, \bar{c})$, and hence also in Aut$(M, \bar{c})$. Hence $H \cap F$ contains all strong automorphisms of $(M, \bar{c})$. It follows that tp$(\bar{b}/M)$ extends to strongly determined type over $\bar{c}$ (with respect to Th(M)). □

An ω-categorical structure M is *smoothly approximated* if there is a chain $M_0 \subset M_1 \subset \ldots$ of finite substructures such that $M = \bigcup\{M_i \colon i \in \omega\}$ and for each i any tuples $\bar{a}$ and $\bar{b}$ from M_i lie in the same Aut(M)-orbit if and only if they lie in the same orbit of the setwise stabiliser of M_i in Aut(M) (see [28] for preliminary results, and [6] or [8] for a general structure theory).

Theorem 5.3. *Let M be a smoothly approximated structure. Then there are $a \in M$ and a non-algebraic $p(x) \in S(M)$ definable almost over a. In particular, M has a strongly determined type over a.*

Proof. Let a realise a non-algebraic type of Th(M). The following theorem is proved in [6] (for the definitions see [6, 8]).

There is M^+, an expansion of M by definitions, and a sequence $a_0, \ldots, a_n \in M^+ \cap \mathrm{dcl}(a)$ such that $a_n = a$ and for each i one of the following holds:

(a) $a_i \in \mathrm{acl}(a_{i-1})$;

(b) there exists $i' < i$, an $a_{i'}$-definable projective Lie geometry J_a^i such that $a_i \in J_a^i$ and $(M^+, a_{i'})$ is a symmetric extension of J_a^i;
(c) there are i', i'' with $i' < i'' < i$ and an $a_{i''}$-definable affine or quadratic geometry (V^i, A^i) such that $a_i \in A^i$, $(M^+, a_{i''})$ is a symmetric extension of (V^i, A^i), and V^i has projectivisation $J_a^{i'}$.

We now may assume that $\mathrm{acl}(a) = \mathrm{acl}(a_{n-1})$ and $a' := a_{n-1}$ is not algebraic over $a'' := a_{n-2}$. To obtain the setting of Lemma 5.2, we regard a'' as a constant. Let N be an a''-definable projective, affine, or quadratic Lie geometry arising at Step $n-1$ of the above construction (with $a' \in N$). We apply Lemma 5.2 with $\bar{d} = \mathrm{dcl}(a) \cap N$ and $\bar{c} = a$. Choose a pair $(p(x), q_p(y))$ such that $q_p(y)$ is a non-algebraic type over $\mathrm{dcl}(a) \cap N$ with respect to the structure N and $p(x)$ is a type of (M, a) such that the algebraic closure of any element realising $q_p(y)$ contains an element realising p. (Since $a \in \mathrm{acl}(a')$, and N is a''-definable and a'' is regarded as a constant, *every* element of $\mathrm{tp}(a/a'')$ is algebraic over some element of N, so it is not hard to find p, q_p. In particular, we may choose p so that a, a' realise respectively the restrictions of p, q_p to a''. Observe that our condition on algebraic closure is stronger than the assumption in Lemma 5.2.)

It follows from Section 14 of [8] and the proof of Theorem 3.1 of [23] that (M^+, a'') is G-finite (indeed, any smoothly approximable structure is). So, if $q_p(y)$ extends to a strongly determined type, we are in the situation of Lemma 5.2.

It remains to check that $(N, \bar{d})$ has strongly determined types. This can be done by inspection of all possible cases of N (see [6, 8]). One of the most interesting possibilities is the case of the affine space of a vector space over a finite field, possibly endowed with a bilinear or quadratic form. Here note that fixing a' we define on N the structure of the corresponding vector space. Now a strongly determined type can be easily obtained (in unstable cases we define it by orthogonality).

Finally, Lemma 5.2 ensures that p extends to a non-algebraic strongly determined type over aa''. Since $a'' \in \mathrm{dcl}^{\mathrm{eq}}(a)$, this gives a strongly determined type over a. □

Remark 5.4. (1) The above argument proves slightly more, namely that in a smoothly approximated structure M, for any non-algebraic 1-type p and $a \in M$ realising p, there is a strongly determined non-algebraic extension of p over a (cf. Proposition 2.12).
(2) The affine space of a symplectic vector space does not have strongly determined types over $\emptyset$. Indeed, let V be a symplectic vector space over a field K, the bilinear form of V be denoted by $\langle -, - \rangle$, and (V, A) be the corresponding affine space (so A is a sort and there is a 0-definable regular action of V on A). Let $u \in V \setminus \{0\}$ and $a, b \in A$ satisfy $\langle u, b - a \rangle = 1$. Then the pairs (a, u) and (b, u) have the same strong type over $\emptyset$ (as $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$). If c realises a type over (V, A) definable over $\emptyset$ then $\mathrm{tp}(a, u, c) = \mathrm{tp}(b, u, c)$. In particular, $\langle u, c - a \rangle = \langle u, c - b \rangle$ and $0 = \langle u, c - a \rangle - \langle u, c - b \rangle = \langle u, b - a \rangle = 1$, a contradiction. The authors are grateful to E. Hrushovski for this observation.
(3) We conjecture that

> any smoothly approximated structure has an expansion by finitely many constants which admits strongly determined types over $\emptyset$.

This conjecture seems difficult and is connected with the problem of the small index property for smoothly approximated structures. Chowdhury et al. have shown in [9] that the affine covers of Lie geometries have the small index property. In the process they prove a kind of joint embedding property over sufficiently saturated 'envelopes' [9, Proposition 6.7]. It follows from Proposition 2.13 above that the conjecture holds for affine covers of Lie geometries.

In the rest of the section we refine Lemma 5.2 in the case of finite covers. Then we give a tight structure theorem for superlinked finite covers of certain ω-categorical structures W, where W is assumed to admit strongly determined types over $\emptyset$ (and have certain other properties). These last two results are also given in [17], but we include them for completeness.

The definitions below are standard (for example, see [14]). Let W be an ω-categorical structure and $\pi : C \to W$ be a surjection, put $C(w) := \pi^{-1}(w)$ (where $w \in W$), and suppose that the sets $C(w)$ are finite and form an $\mathrm{Aut}(C)$-invariant partition of C. The structure C is a *finite cover* of W under the map π if the image of the induced map $\tau : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$ (which is called the *restriction mapping*) is $\mathrm{Aut}(W)$. The kernel of τ is called the *kernel* of C (denoted by $\mathrm{Ker}(C)$). The cover C is *superlinked* if its kernel is finite. If the kernel of a cover is trivial then the cover is called *trivial*. We say that C is *split* if it has an expansion which is a trivial cover of W under π. If C is split and the corresponding trivial cover is obtained from C by adding unary predicates which form a partition of C such that each class intersects each fibre in a singleton, then we say that C is *strongly split*. It is clear that the permutation group $(\mathrm{Aut}(C), C)$ of a strongly split cover is easily reconstructed by the groups $(\mathrm{Aut}(W), W)$ and $(\mathrm{Ker}(C), C)$. This explains the content of Theorem 5.6 below.

In the above setting, we use a subscript W or C (as in $S_C(C)$, $\mathrm{tp}_W(\bar{a})$) to indicate whether a type is considered with respect to $\mathrm{Th}(W)$ or $\mathrm{Th}(C)$.

Note that $C \cup W$, with the fibre structure given from π, can be regarded as a symmetric extension of W. Furthermore, for any type $p(\bar{x})$ of the structure C there is a type $q_p(\bar{y})$ of the structure W such that for any $\bar{a} \models q_p$ there is $\bar{b} \models p$ with $\bar{b} \in \mathrm{acl}_{C \cup W}(\bar{a})$. These observations yield the following application of Proposition 5.1 to finite covers.

Theorem 5.5. *Let W be an ω-categorical structure such that* $\mathrm{Aut}^o(W)$ *has finite index in* $\mathrm{Aut}(W)$. *Let ρ be a strongly determined n-type over $\emptyset$ of* $\mathrm{Th}(W)$ *and $\pi : C \to W$ be a finite cover of W. For $\bar{a} \models \rho_\emptyset$ let $\bar{b}$ be an enumeration of $C(\bar{a}) := \pi^{-1}(\bar{a})$. Then* $\mathrm{tp}_C(\bar{b}/\emptyset)$ *extends to a strongly determined type over $\emptyset$.*

Proof. We work in an ω_1-saturated elementary extension of C. Choose $\bar{b}' \models \mathrm{tp}_C(\bar{b})$ such that $\bar{a}' := \pi(\bar{b}') \models \rho_W$. Define

$G := \{g \in \mathrm{Aut}(C) : g \cup \mathrm{id}_{\bar{b}'}$ is $\mathrm{Th}(C)$-elementary$\}$,

$F := \{g \in \mathrm{Aut}(C) : g \cup \mathrm{id}_{\bar{a}'}$ is $\mathrm{Th}(C)$-elementary$\}$,

$H := \{g \in \mathrm{Aut}^o(W) : g \cup \mathrm{id}_{\bar{a}'}$ is $\mathrm{Th}(W)$-elementary$\}$,

$K := \mathrm{Ker}(C)$.

Note that G is a closed subgroup of $\mathrm{Aut}(C)$, as elementary maps are finitely determined. Also, $H = \mathrm{Aut}^o(W)$, by the choice of $\bar{a}'$. Furthermore $G \leqslant F$.

We claim that any automorphism in K extends to an elementary map $C\bar{a}' \to C\bar{a}'$, that is, $K \leqslant F$. To see this it suffices to show that for any finite tuple $\bar{w}$ in W there exists a tuple $\bar{a}_1$ of elements of W with $\mathrm{tp}_C(\bar{a}'/C(\bar{w})) = \mathrm{tp}_C(\bar{a}_1/C(\bar{w}))$. But $\mathrm{tp}_C(\bar{a}'/C(\bar{w}))$ is determined by $\mathrm{tp}_W(\bar{a}'/\bar{w}')$ for some finite tuple $\bar{w}'$ (by openness of the restriction mapping – see Lemmas 5(b), 7 from [24]), so we may choose any $\bar{a}_1 \models \rho_{\bar{w}'}$.

By Proposition 5.1 the type ρ extends to a strongly determined type of C and now we may assume that $\bar{a}'$ realises its restriction over C. Then the group F contains all strong automorphisms of C and hence induces on W a group containing H. Since $K \leqslant F$, it follows that the group F contains all automorphisms of C which induce elements of H. Since $|\mathrm{Aut}(W):H|$ is finite, it follows that F has finite index in $\mathrm{Aut}(C)$. Clearly $|F:G|$ is finite, so $|\mathrm{Aut}(C):G|$ is finite, and hence G contains all strong automorphisms of C. It follows that $\mathrm{tp}_C(\bar{b}'/C)$ is definable almost over $\emptyset$.

In the next result we assume that $\mathrm{acl}(A) = \mathrm{dcl}(A)$ for all $A \subset W$ and that W has weak elimination of imaginaries: that is, for every $c \in W^{\mathrm{eq}}$ there is a finite $A \subset W \cap \mathrm{acl}^{\mathrm{eq}}(c)$ such that $c \in \mathrm{dcl}^{\mathrm{eq}}(A)$. Many familiar ω-categorical structures satisfy these conditions. The conditions imply that for all finite $X \subset W$, the group $\mathrm{Aut}(W/X)$ does not have proper closed subgroups of finite index (see Lemma 1.3 of [16]). Thus any type which is definable almost over X is definable over X. The following result is similar to Corollary 2.4 of [14]. The proof is rather different, and we are not assuming that $\mathrm{Aut}\, C = \mathrm{Aut}^o(C)$ – rather, we put strong assumptions on W.

Theorem 5.6. *Assume that W is an ω-categorical structure with* $\mathrm{acl}(A) = \mathrm{dcl}(A)$ *for all $A \subset W$, such that* $\mathrm{Th}(W)$ *has weak elimination of imaginaries and admits strongly determined types. Let $\pi : C \to W$ be a superlinked finite cover of W and E be the finest 0-definable finite equivalence relation on C. Then*
(i) *each E-class meets each fibre of π in at most one point,*
(ii) *if W is transitive then C is strongly split.*

Proof. Let $\bar{a}$ be a tuple from W such that every $\alpha \in \mathrm{Aut}(C/W)$ fixing $\bar{b}$ pointwise is trivial, where $\bar{b}$ is an enumeration of $\pi^{-1}(\bar{a})$. It follows from weak elimination of imaginaries and openness of the restriction mapping that for any $c \in C$ the type of $(c, \bar{b})$ over $\mathrm{acl}(c, \bar{b}) \cap W$ implies $\mathrm{tp}_C(c\bar{b}/W)$. By the choice of $\bar{b}$, if $\mathrm{tp}_C(c\bar{b}/W) = \mathrm{tp}_C(c'\bar{b}/W)$ then $c = c'$. It follows that $c \in \mathrm{dcl}(\bar{b} \cup (\mathrm{acl}(c\bar{b}) \cap W))$. Since $\mathrm{dcl} = \mathrm{acl}$ in W,

$$\mathrm{acl}(c\bar{b}) \cap W \subseteq \mathrm{acl}(\pi(c\bar{b})) \cap W = \mathrm{dcl}(\pi(c\bar{b})) \cap W \subseteq \mathrm{dcl}(\pi(c)\bar{b}),$$

so $c \in \mathrm{dcl}(\bar{b} \cup \{\pi(c)\})$.

By Theorem 5.5 there exists a strongly determined type ρ (of $\mathrm{Th}(C)$) which extends $\mathrm{tp}(\bar{b}/\emptyset)$. For distinct c and c' satisfying $\pi(c) = \pi(c')$ consider the types ρ_c and $\rho_{c'}$. We may assume that $\bar{b} \models \rho_{cc'}$. Since $c, c' \in \mathrm{dcl}(\bar{b} \cup \{\pi(c)\})$, the types of $c\bar{b}$ and $c'\bar{b}$ are different. Thus if $\alpha \in \mathrm{Aut}(C)$ sends c to c', then $\alpha(\rho_c) \neq \rho_{\alpha(c)}$. Hence $\alpha \notin \mathrm{Aut}^o(C)$ and the elements c and c' lie in distinct E-classes, so (i) holds.

To prove (ii), assume that W is transitive. Then by our assumption on algebraic closure, there are no non-trivial 0-definable finite equivalence relations on W. Hence if W is transitive then each E-class meets each fibre. It follows that by adding unary predicates for the E-classes we get a trivial cover of W, that is, one with trivial kernel. $\square$

6. Summary

We try here to summarise the main motivation and results of the paper.

Regarding motivation, we are looking at a very weak notion of independence over a base set A, which generalises the notion of definable type (almost) over A. More specifically, we are trying to develop multiplicity in an unstable setting – hence the distinction between strongly determined types and Shelah's notion of non-splitting extension. In Section 4.3 the class of *FM-theories* has been identified, and this class resembles that of ω-stable theories, the usual context for multiplicity. The class of *FM*-theories includes that of finitely homogeneous structures admitting strongly determined types, a very intractable class with many unstable members, and we expect that our methods will have further applications here (see Section 4.3). The notions also have applications to the structure of finite covers (see Section 5).

The associated notion of independence generalises forking, but does not satisfy the axioms of symmetry, extension, and local character. It may be that by restricting the class of theories to those satisfying the former two axioms a theory of independence can be obtained similar to non-forking (for example, in Section 4.2 we show that the stable Finite Equivalence Relation Theorem holds under some general assumptions). Unfortunately, we do not know how wide this class is. Problem 4.6 is a starting point in this direction. It seems that without a symmetry axiom, there will not be a deep theory of independence.

We have proved results showing existence of strongly determined types for a wide range of theories. In some cases (e.g. weakly o-minimal theories in Theorem 2.6, and those satisfying Corollary 2.4) the strongest possible result has been shown, that all types over all sets have strongly determined extensions. In other cases (such as those considered in Section 2.5) it has been shown that all types over $\emptyset$ have strongly determined extensions. For P-minimal theories, it is merely shown that there is a strongly determined type over $\emptyset$ (Theorem 2.10), and for smoothly approximable theories and certain binary homogeneous structures, the existence of a strongly determined type is proved over a singleton (Theorem 5.3 and Proposition 2.12).

We would expect there to be applications for theories such that every finite set A is contained in a finite set A' such that the theory admits strongly determined types over A'. It seems likely that this holds for smoothly approximable structures, but Theorem 5.3 is not quite strong enough to yield this. Note that as in Example 3.5, it can happen that a theory admits strongly determined types over $\emptyset$ but not over some non-trivial constant expansion. The above condition rules such examples out, as it allows us to remedy the situation by adding constants.

There is hope for further development of these notions in restricted classes of finitely homogeneous structures, such as the class of *finitely constrained* homogeneous structures: a homogeneous structure in a finite relational language is *finitely constrained* if there is a finite set S of finite L-structures such that the finite structures which embed in M are precisely those not admitting a member of S as a substructure.

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